FINITE DIFFERENCE AND PDE

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Numerical methods: properties

Finite differences →

- time-dependent PDEs
-> robust, simple concept, easy to parallelize, regular grids, explicit method

Finite elements →

- static and time-dependent PDEs
-> implicit approach, matrix inversion, well founded, irregular grids, more complex algorithms, engineering problems

Finite volumes →

- time-dependent PDEs
-> robust, simple concept, irregular grids, explicit method

Other numerical methods

Particle-based methods →

- lattice gas methods
- molecular dynamics
- granular problems
- fluid flow
- earthquake simulations

-> very heterogeneous problems, nonlinear problems

Boundary element methods →

- problems with boundaries (rupture)
- based on analytical solutions
- only discretization of planes

-> good for problems with special boundary conditions (rupture, cracks, etc)

***Pseudo*spectral methods** →

- orthogonal basis functions, special case of FD
- spectral accuracy of space derivatives
- wave propagation, ground penetrating radar

-> regular grids, explicit method, problems with strongly heterogeneous media

What is a finite difference?

Common definitions of the derivative of f(x):

$$\partial_x f = \lim_{dx \to 0} \frac{f(x+dx) - f(x)}{dx}$$

$$\partial_x f = \lim_{dx \to 0} \frac{f(x) - f(x-dx)}{dx}$$

$$\partial_x f = \lim_{dx \to 0} \frac{f(x+dx) - f(x-dx)}{2dx}$$

These are all correct definitions in the limit dx->0.

But we want dx to remain **FINITE**

What is a finite difference?

The equivalent ***approximations*** of the derivatives are:

$$\partial_x f^+ \approx \frac{f(x+dx)-f(x)}{dx}$$

forward difference

$$\partial_x f^- \approx \frac{f(x)-f(x-dx)}{dx}$$

backward difference

$$\partial_x f \approx \frac{f(x+dx)-f(x-dx)}{2dx}$$

centered difference

The **big** question:

How good are the FD approximations?

$$\approx \neq ==$$

This leads us to Taylor series....

Taylor Series

Taylor series are expansions of a function f(x) for some finite distance dx to f(x+dx)

$$f(x \pm dx) = f(x) \pm \mathrm{dx} f'(x) + \frac{dx^2}{2!} f''(x) \pm \frac{dx^3}{3!} f'''(x) + \frac{dx^4}{4!} f''''(x) \pm \ldots$$

What happens, if we use this expression for

$$\partial_x f^+ \approx \frac{f(x+dx) - f(x)}{dx}$$

?

Taylor Series

... that leads to :

$$\frac{f(x+dx)-f(x)}{dx}=\frac{1}{dx}\left[\mathrm{dx}f^{'}(x)+\frac{dx^2}{2!}f^{''}(x)+\frac{dx^3}{3!}f^{'''}(x)+\ldots\right]$$

$$=f^{'}(x)+O(dx)$$

The error of the first derivative using the *forward* formulation is *of order dx*.

Is this the case for other formulations of the derivative? Let’s check!

Taylor Series

... with the *centered* formulation we get:

$$\frac{f(x+dx/2)-f(x-dx/2)}{dx}=\frac{1}{dx}\left[\mathrm{dx}f^{'}(x)+\frac{dx^3}{3!}f^{'''}(x)+\ldots\right]$$

$$=f^{'}(x)+O(dx^2)$$

The error of the first derivative using the centered approximation is *of order* dx^2.

This is an **important** results: it DOES matter which formulation we use. The centered scheme is more accurate!

Higher order operators

$$*a \,|\qquad f(x-2dx) \approx f - (2dx)f' + \frac{(2dx)^2}{2!}f'' - \frac{(2dx)^3}{3!}f'''$$

$$*b \,|\qquad f(x-dx) \approx f - (dx)f' + \frac{(dx)^2}{2!}f'' - \frac{(dx)^3}{3!}f'''$$

$$*c \,|\qquad f(x+dx) \approx f + (dx)f' + \frac{(dx)^2}{2!}f'' + \frac{(dx)^3}{3!}f'''$$

$$*d \,|\qquad f(x+2dx) \approx f + (2dx)f' + \frac{(2dx)^2}{2!}f'' + \frac{(2dx)^3}{3!}f'''$$

... again we are looking for the coefficients a,b,c,d with which the function values at x±(2)dx have to be multiplied in order to obtain the interpolated value or the first (or second) derivative!

... Let us add up all these equations like in the previous case ...

Problems: Stability

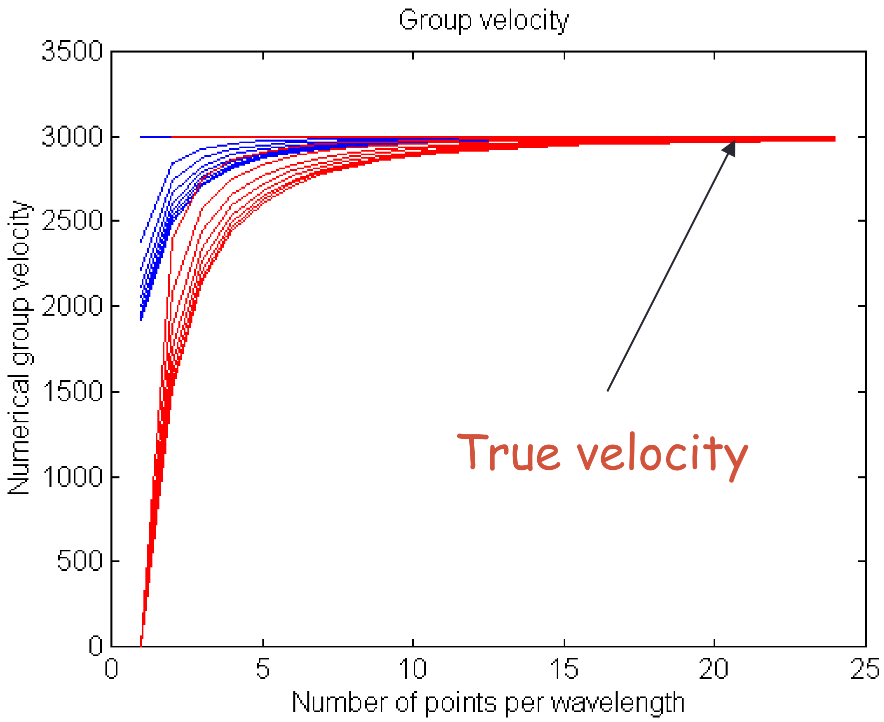
$$p(t+dt)=\frac{c^2dt^2}{dx^2}\left[p(x+dx)-2p(x)+p(x-dx)\right]$$
$$+2p(t)-p(t-dt)+sdt^2$$

Stability: Careful analysis using harmonic functions shows that a stable numerical calculation is subject to special conditions (conditional stability). This holds for many numerical problems. (Derivation on the board).

$$c\frac{dt}{dx}\leq\varepsilon\approx 1$$

Problems: Dispersion

$$p(t+dt) = \frac{c^2 dt^2}{dx^2}\left[p(x+dx) - 2p(x) + p(x-dx)\right] + 2p(t) - p(t-dt) + sdt^2$$

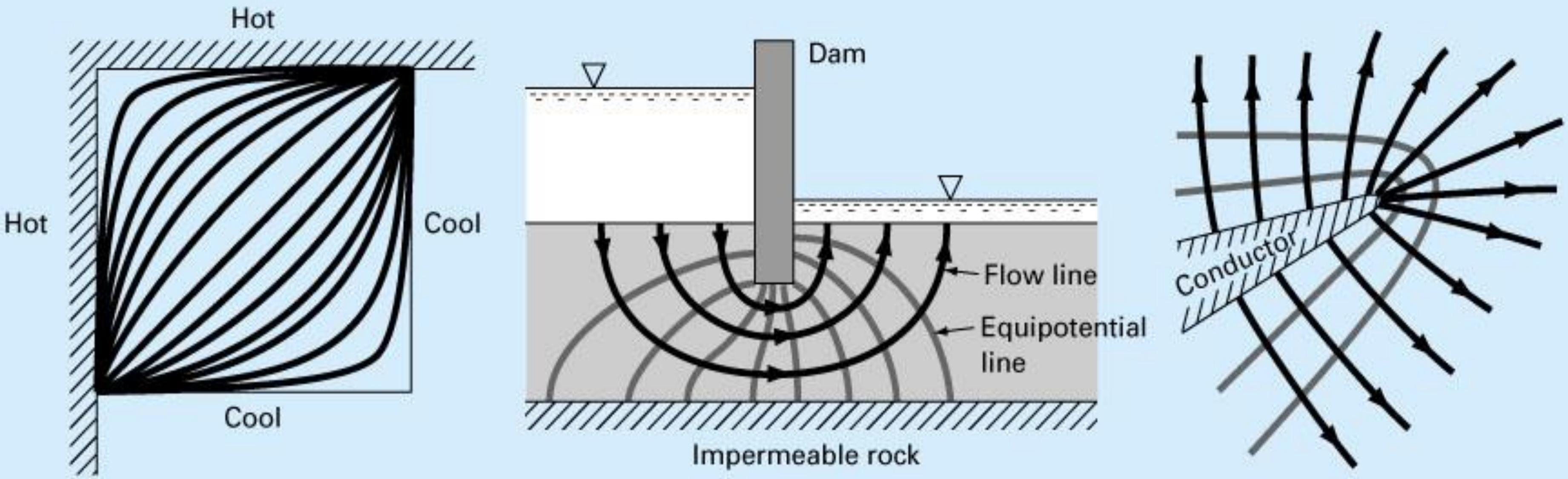


Dispersion: The numerical approximation has artificial dispersion, in other words, the wave speed becomes frequency dependent (Derivation in the board).
You have to find a frequency bandwidth where this effect is small.
The solution is to use a sufficient number of grid points per wavelength.

Finite Differences - Summary

- Conceptually the most simple of the numerical methods and can be learned quite quickly
- Depending on the physical problem FD methods are conditionally stable (relation between time and space increment)
- FD methods have difficulties concerning the accurate implementation of boundary conditions (e.g. free surfaces, absorbing boundaries)
- FD methods are usually explicit and therefore very easy to implement and efficient on parallel computers
- FD methods work best on regular, rectangular grids

Partial Differential Equations



(a) (b) (c)

PERSAMAAN DIFERENSIAL PARSIAL

- Persamaan Umum

$$a\frac{\partial^2\theta}{\partial x^2}+b\frac{\partial^2\theta}{\partial x\partial y}+c\frac{\partial^2\theta}{\partial y^2}+d\frac{\partial\theta}{\partial x}+e\frac{\partial\theta}{\partial y}+f\theta+g=0$$

- Menyatakan bagaimana variabel tak bebas ∅ berubah terhadap variabel bebas x,y. Disini a,b,c,d,e,f, dan g mungkin merupakan fungsi dari ∅

Jenis2 PDP

- Ditentukan oleh harga b^2-4ac

  < 0, eliptic

  = 0, parabolic

  > 0, hyperbolic

- Adveksi…
- Difusi….
- Gelombang…

JENIS-JENIS PDP

A. Persamaan Differensial Parabolik

➔ Biasanya merupakan persamaan yang tergantung pada waktu (tidak permanen) dan penyelesaiannya memerlukan kondisi awal dan batas. Persamaan parabolik paling sederhana adalah perambatan panas.

$$\frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2}$$

Penyelesaian dari persamaan di atas adalah mencari temperatur T untuk nilai x pada setiap waktu t.

B. Persamaan Differensial Eliptik

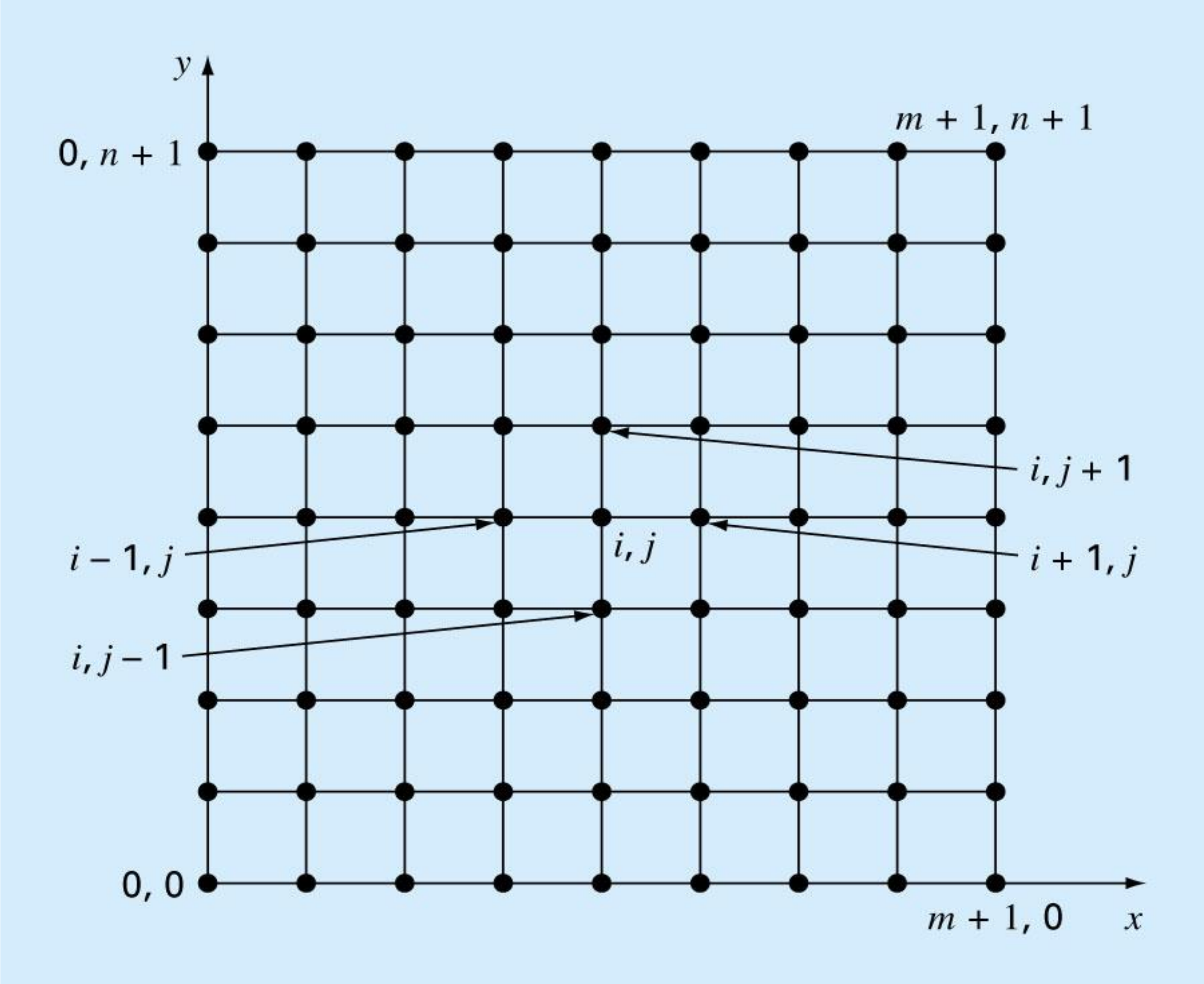
➔ Biasanya berhubungan dengan masalah kesetimbangan atau kondisi permanen (tidak tergantung waktu) dan penyelesaiannya memerlukan kondisi batas di sekeliling daerah tinjauan. Seperti aliran air tanah di bawah bendungan dan karena adanya pemompaan, defleksi plat akibat pembebanan, dsb.

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$$

C. Persamaan Differensial Hiperbolik

➔ Biasanya berhubungan dengan getaran atau permasalahan dimana terjadi diskontinue dalam waktu, seperti gelombang kejut yang terjadi discontinue dalam kecepatan, tekanan dan rapat massa.

$$\frac{\partial^2 U}{\partial t^2} = C^2 \frac{\partial^2 U}{\partial x^2}$$


y
0, n + 1
m + 1, n + 1
i, j + 1
i − 1, j
i, j
i + 1, j
i, j − 1
0, 0
m + 1, 0
x

The Laplacian Difference Equations/

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Laplace Equation

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2}$$

$O[\Delta(x)^2]$

$$\frac{\partial^2 T}{\partial y^2} = \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta y^2}$$

$O[\Delta(y)^2]$

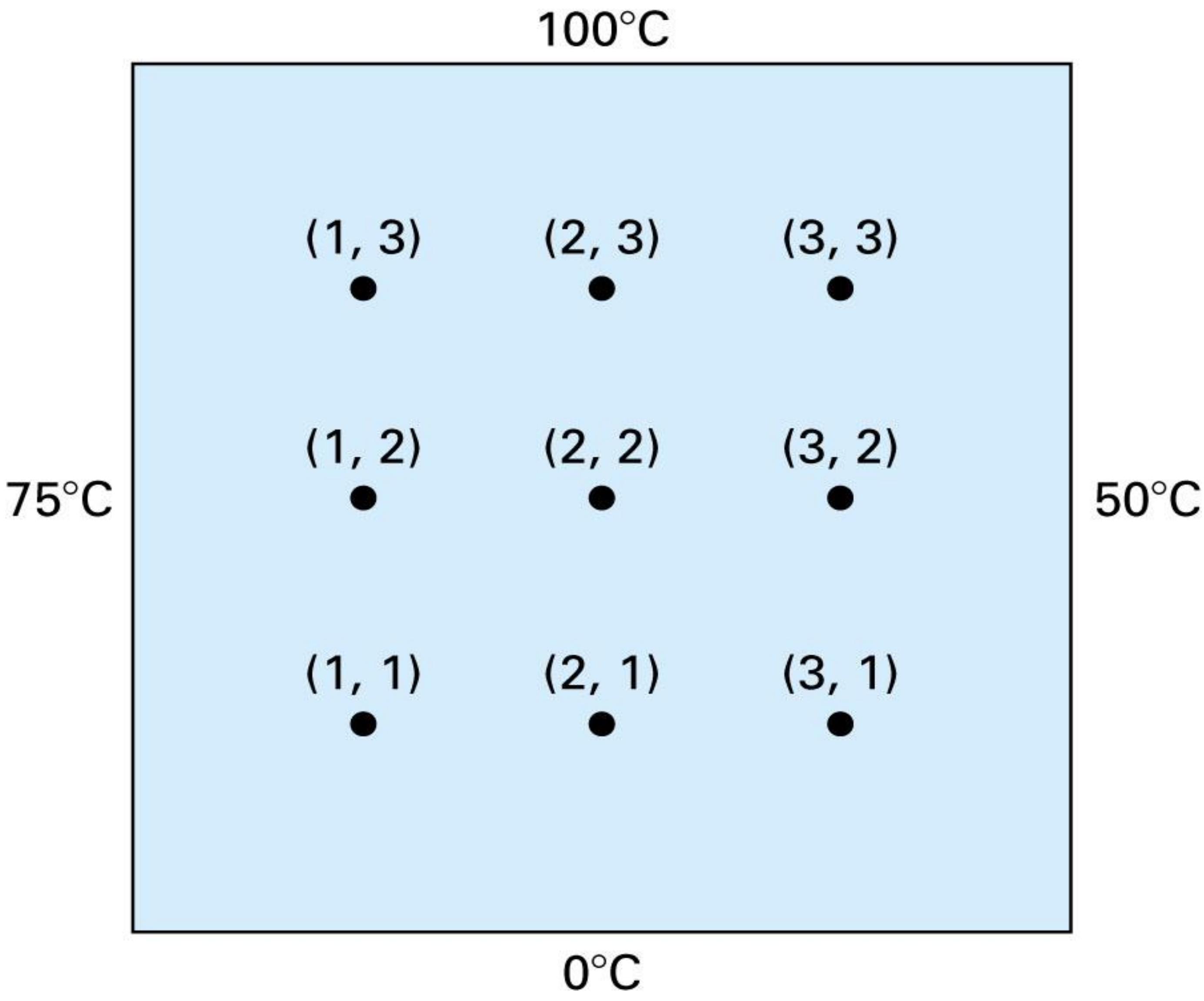
$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta y^2} = 0$$

$$\Delta x = \Delta y$$

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$$

Laplacian difference equation.

Holds for all interior points


100°C
(1, 3)
(2, 3)
(3, 3)
(1, 2)
(2, 2)
(3, 2)
75°C
50°C
(1, 1)
(2, 1)
(3, 1)
0°C

- In addition, boundary conditions along the edges must be specified to obtain a unique solution.
- The simplest case is where the temperature at the boundary is set at a fixed value, *Dirichlet boundary condition*.
- A balance for node (1,1) is:

$$T_{21} + T_{01} + T_{12} + T_{10} - 4T_{11} = 0$$

$$T_{01} = 75$$

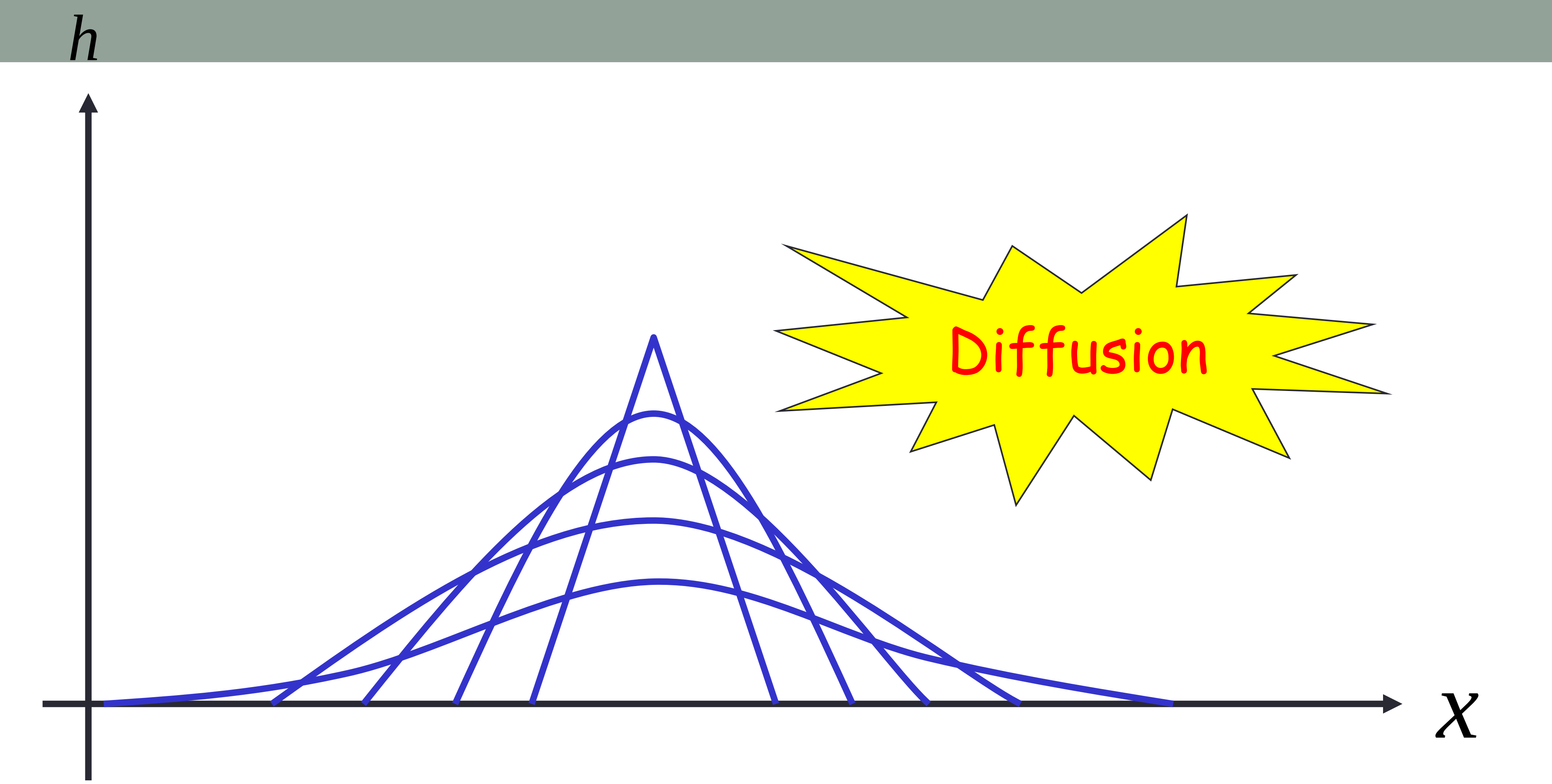
$$T_{10} = 0$$

$$-4T_{11} + T_{12} + T_{21} = 0$$

- Similar equations can be developed for other interior points to result a set of simultaneous equations.

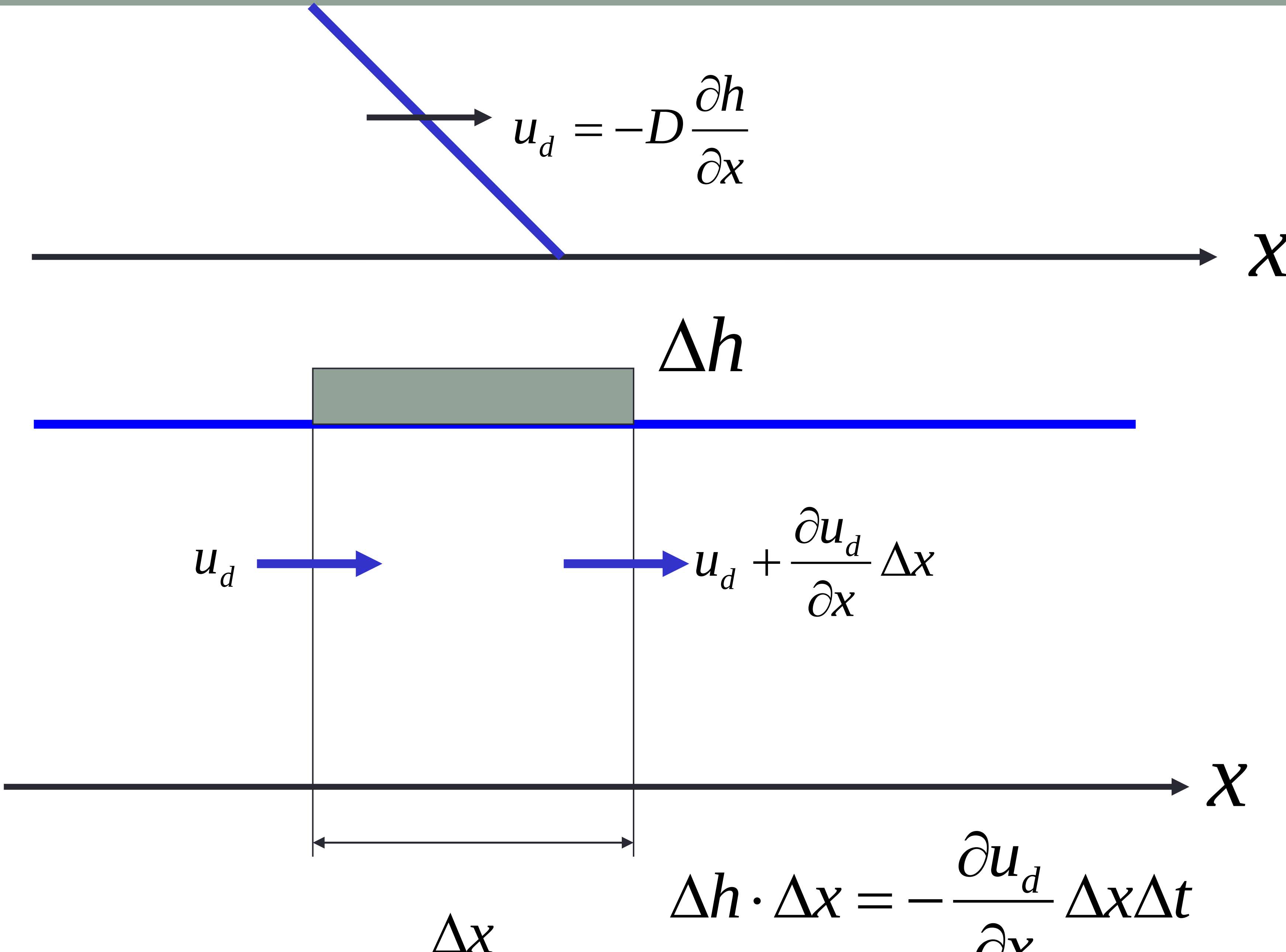
- The result is a set of nine simultaneous equations with nine unknowns:

$$
\begin{array}{llllllllll}
4T_{11} & -T_{21} & & -T_{12} & & & & & & =75 \\
-T_{11} & +4T_{21} & -T_{13} & & -T_{22} & & & & & =0 \\
 & -T_{21} & +4T_{31} & & & -T_{32} & & & & =50 \\
-T_{11} & & & +4T_{12} & -T_{22} & & -T_{13} & & & =75 \\
 & -T_{21} & & -T_{12} & +4T_{22} & -T_{32} & & -T_{23} & & =0 \\
 & & -T_{31} & & -T_{22} & +4T_{32} & & & -T_{33} & =50 \\
 & & & -T_{12} & & & +4T_{13} & -T_{23} & & =175 \\
 & & & & -T_{22} & & -T_{13} & +4T_{23} & -T_{33} & =100 \\
 & & & & & -T_{32} & & -T_{23} & +4T_{33} & =150
\end{array}
$$



$$\frac{\partial h}{\partial t} = D\frac{\partial^2 h}{\partial x^2}$$

Diffusion Equation



$$\Delta h \cdot \Delta x = -\frac{\partial u_d}{\partial x}\Delta x \Delta t$$

$$\Delta h \cdot \Delta x = -\frac{\partial u_d}{\partial x} \Delta x \Delta t$$

$$\Delta t \rightarrow 0$$

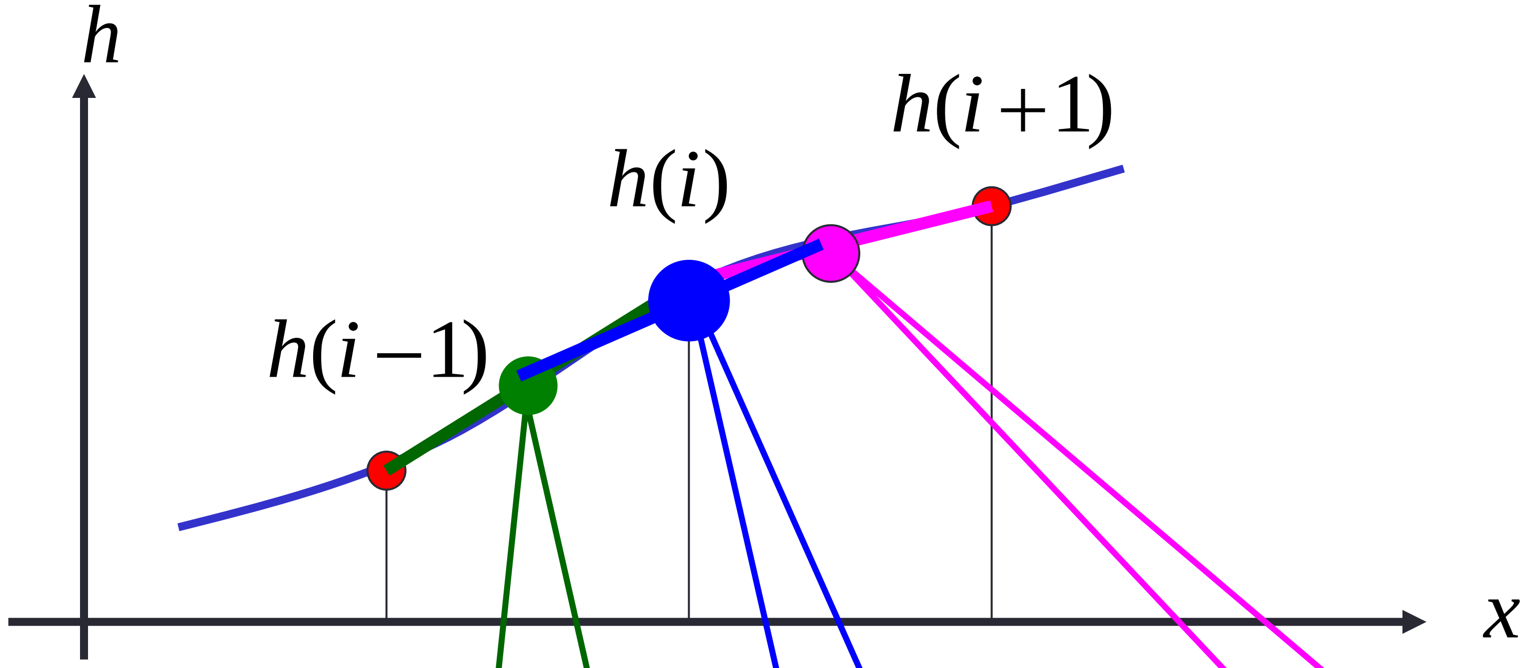
$$\frac{\partial h}{\partial t} = -\frac{\partial u_d}{\partial x}$$

$$u_d = -D \frac{\partial h}{\partial x}$$

$$\frac{\partial h}{\partial t} = D \frac{\partial^2 h}{\partial x^2}$$

Diffusion Equation

Numerical Solution of Diffusion Eq.

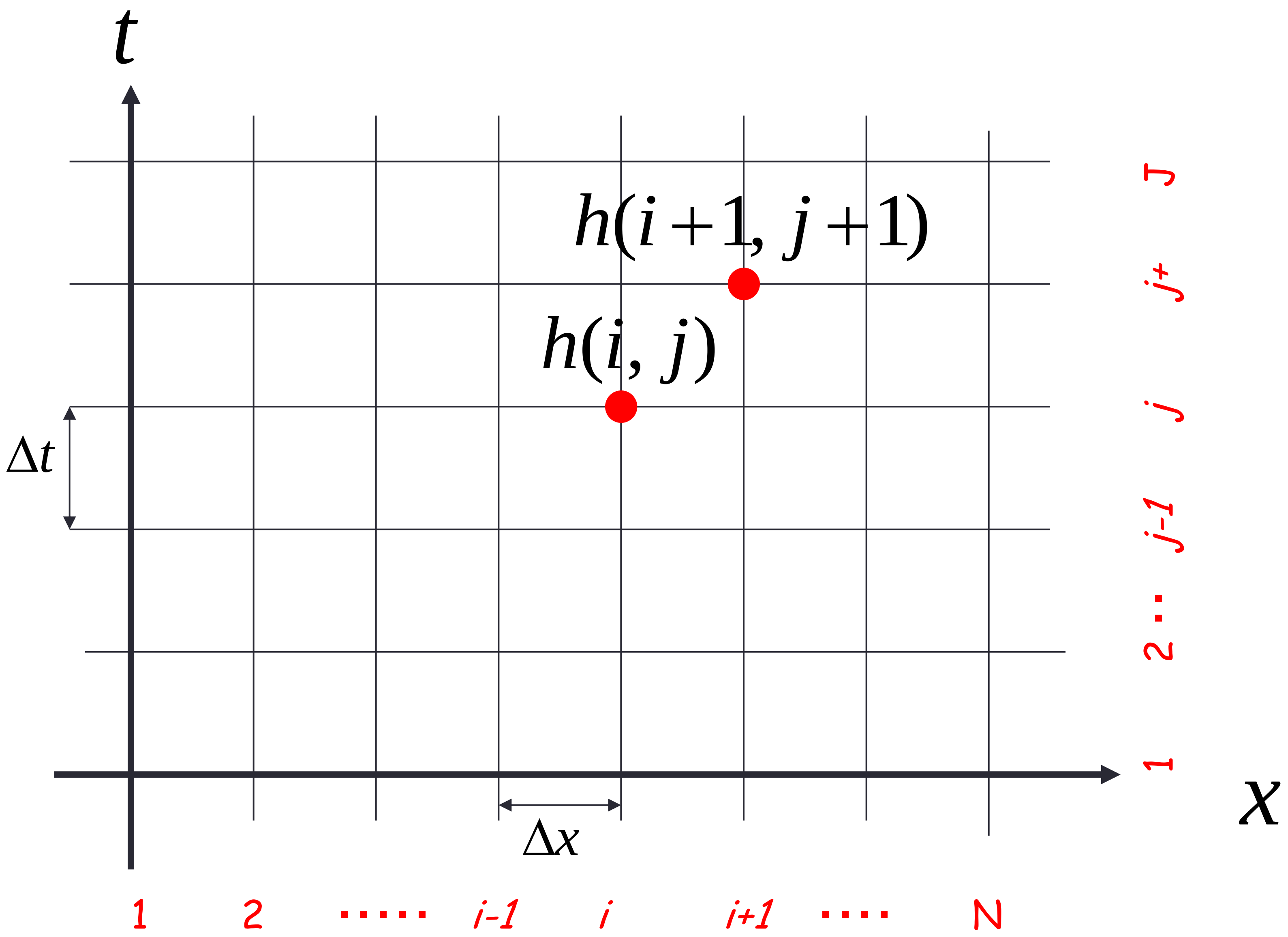


$$\left(\frac{\partial h}{\partial x}\right)_1 = \frac{h(i) - h(i-1)}{\Delta x}$$

$$\left(\frac{\partial h}{\partial x}\right)_2 = \frac{h(i+1) - h(i)}{\Delta x}$$

$$\frac{\partial^2 h}{\partial x^2} = \frac{1}{\Delta x}\left[\left(\frac{\partial h}{\partial x}\right)_2 - \left(\frac{\partial h}{\partial x}\right)_1\right]$$

Numerical Calculation of Diffusion Equation



$$\frac{\partial h}{\partial t} = \frac{h(i, j+1) - h(i, j)}{\Delta t}$$

$$D\frac{\partial^2 h}{\partial x^2} = \frac{D}{\Delta x}\left(\frac{h(i+1, j) - h(i, j)}{\Delta x} - \frac{h(i, j) - h(i-1, j)}{\Delta x}\right)$$

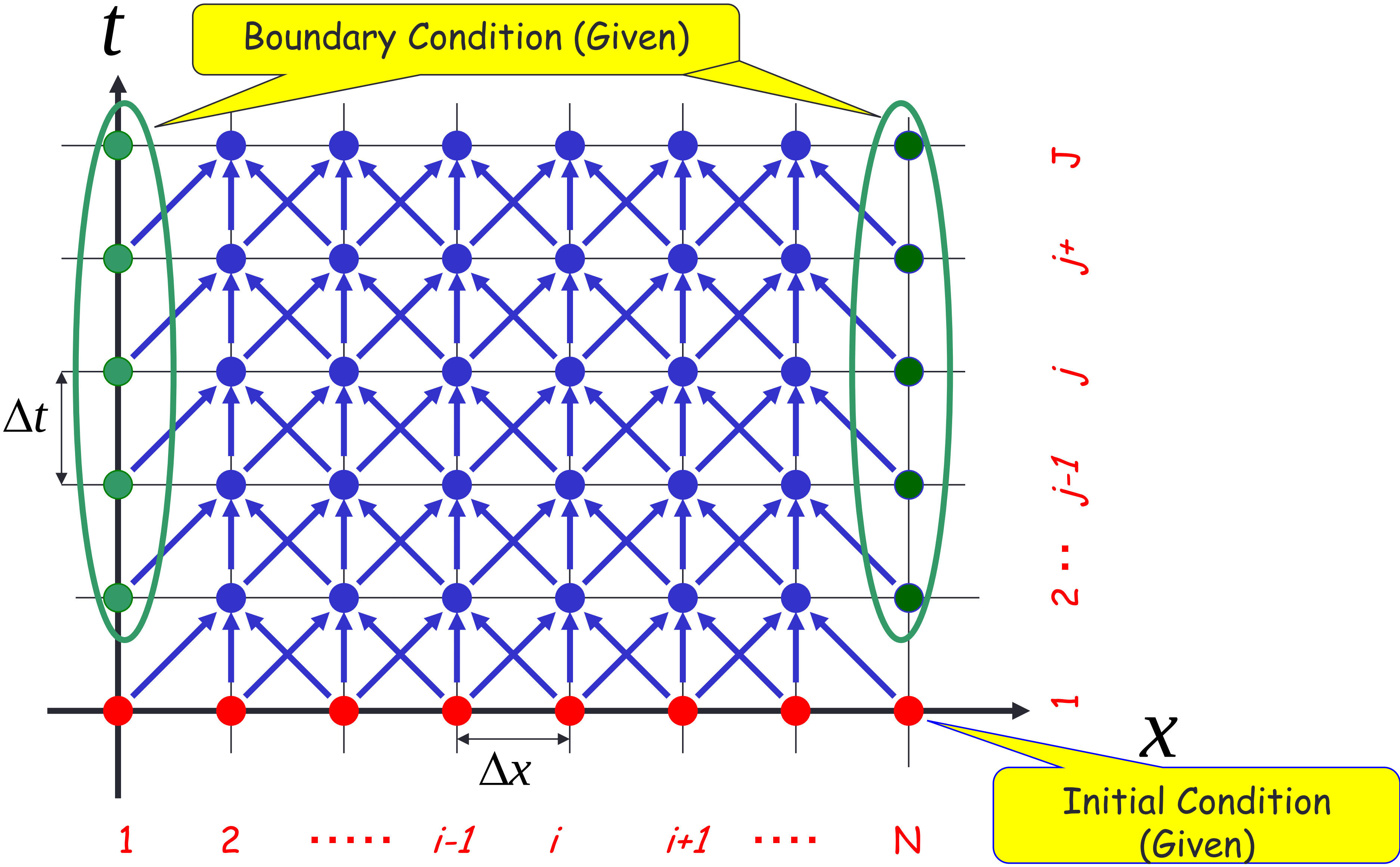
$$= D\frac{h(i+1, j) - 2h(i, j) + h(i-1, j)}{\Delta x^2}$$

Unknown

$$h(i, j+1) = h(i, j) + \frac{D\Delta t}{\Delta x^2}\{h(i+1, j) - 2h(i, j) + h(i-1, j)\}$$

Known

Numerical Solution of Diffusion Equation



$$h(i, j+1) = h(i, j) + \frac{D\Delta t}{\Delta x^2}\{h(i+1, j) - 2h(i, j) + h(i-1, j)\}$$

do $i = 2$,N-1

$$h_{new}(i) = h_{old}(i) + \left[h_{old}(i+1) - 2h_{old}(i) + h_{old}(i-1)\right]\frac{\Delta t}{\Delta x^2}D$$

end do

do $i = 2$,N-1

$$h_{old}(i) = h_{new}(i)$$

end do

Contoh

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4x^2, \quad y(1) = 1, \quad y(2) = 6$$

- Cari solusinya dengan step size 0,2
- Jumlah titik solusi n=((2-1)/0,2)-1=4
- Kita dapatkan 4 persamaan, satu untuk tiap titik yang dicari.

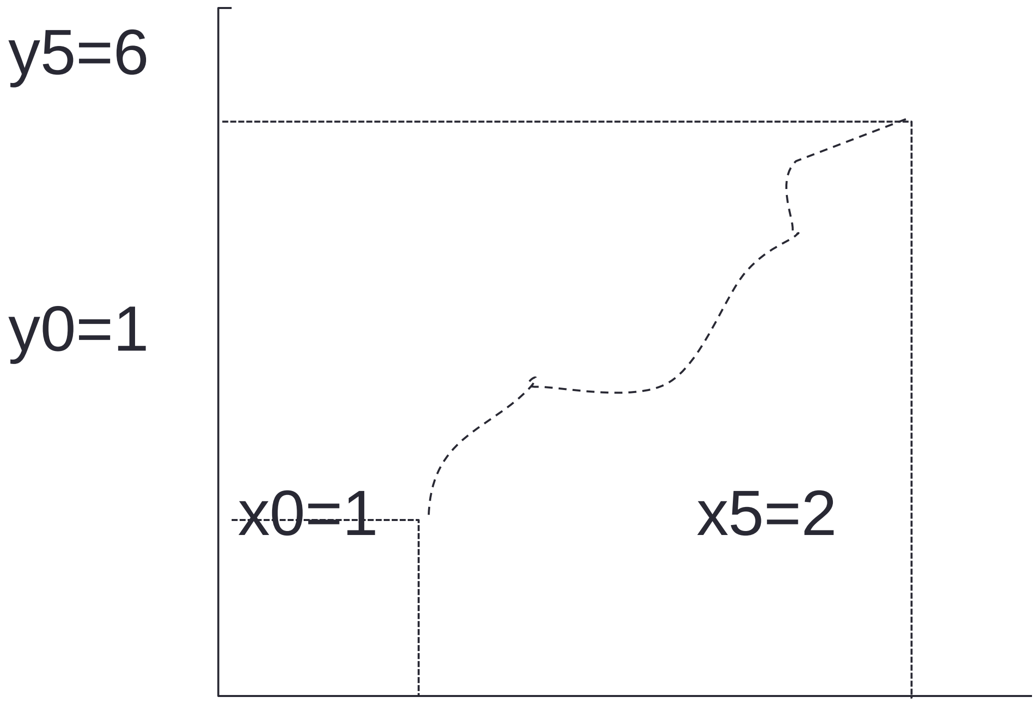
Penyelesaian dengan Beda Hingga

Persamaannya:

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + 3\frac{y_{i+1} - y_{i-1}}{2h} + 2y_i = 4x_i^{\;2}$$

Buat persamaan untuk semua titik, mulai dari i=1, hingga i=4

Domain Solusi



Penyelesaian dg Finite Diff.

- Dengan h = 0,2

$$32{,}5y_{i+1} - 48y_i + 17{,}5y_{i-1} = 4x_i^{\,2}$$

- Buat persamaan untuk semua titik, mulai dari i=1, hingga i=4
- Masukkan nilai-nilai x1=1,2 hingga x4 =1,8 dan kondisi batas y0 = 1 dan y5 = 6

Kondisi batas

- Dirichlet atau fixed boundary,
  misal $C(0) = Co$
- Neuman atau natural boundary,
  misal $dC/dx = 0$
- Robin/ Cauchy boundary condition,
  misal $dC/dx + C = 0$

- Penerapan dalam finite difference dengan menambahkan imaginary node

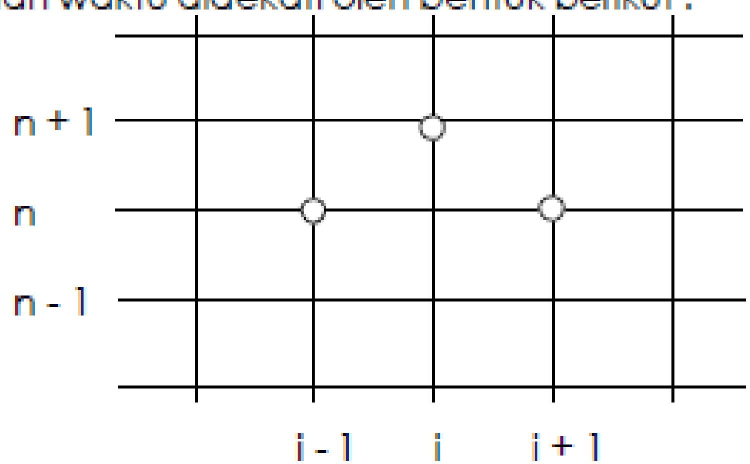
penyelesaian persamaan parabolik dengan skema eksplisit

$$\frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(6.1)}$$

dengan :

T	=	temperatur
K	=	koefisien konduktivitas
t	=	waktu
x	=	jarak

Pada skema eksplisit, variabel pada waktu *n+1* dihitung berdasarkan variabel pada waktu *n* yang sudah diketahui. Dengan menggunakan skema seperti di bawah ini, fungsi f(x,t) dan turunannya dalam ruang dan waktu didekati oleh bentuk berikut :



$$f(x, t) = f_i^n$$

$$\frac{\partial f(x,t)}{\partial t} = \frac{f_i^{n+1} - f_i^n}{\Delta t}$$

$$\frac{\partial f^2(x,t)}{\partial t^2} = \frac{f_{i-1}^n - 2f_i^n + f_{i+1}^n}{\Delta t^2}$$

SEHINGGA…

persamaan diferensial,

$$\frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2}$$

ditulis dalam bentuk metode beda hingga,

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = K_i \frac{T_{i-1}^n - 2T_i^n + T_{i+1}^n}{\Delta x^2}$$

atau

$$T_i^{n+1} = T_i^n + K_i \frac{\Delta t}{\Delta x^2}\left(T_{i-1}^n - 2T_i^n + T_{i+1}^n\right)$$

Penyelesaian dilakukan dengan mendiskretisasi suatu persamaan differensial parsial eliptik dengan kondisi batas untuk dapat ditransformasikan ke dalam suatu sistem dari N persamaan dengan N bilangan anu.

Penyelesaian persamaan eliptik dilakukan dengan langkah-langkah berikut ini.

1. Membuat jaringan titik simpul di dalam seluruh bidang yang ditinjau dan batas-batasnya.
2. Pada setiap titik dalam bidang tersebut dibuat turunan-turunannya dalam bentuk beda hingga.
3. Ditulis nilai-nilai fungsi pada semua titik di batas keliling bidang dengan memperhatikan kondisi batas.

Dari persamaan bentuk eliptik berikut :

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$$

laplace ….equation

Sehingga :

$$\frac{\varphi_{i-1,j} - 2\varphi_{i,j} + \varphi_{i+1,j}}{\Delta x^2} + \frac{\varphi_{i,j-1} - 2\varphi_{i,j} + \varphi_{i,j+1}}{\Delta y^2} = 0$$

Untuk $\Delta x = \Delta y$, maka persamaan di atas menjadi :

$$4\varphi_{i,j} - \varphi_{i-1,j} - \varphi_{i+1,j} - \varphi_{i,j-1} - \varphi_{i,j+1} = 0$$

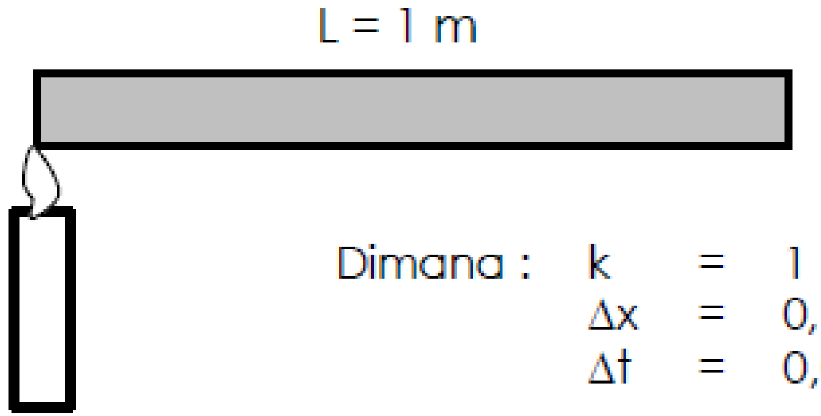
penyelesaian persamaan eliptik

Stabilitas skema eksplisit

Dalam skema eksplisit, T_i^n tergantung pada tiga titik sebelumnya yaitu: T_{i-1}^{n-1}, T_i^{n-1} dan T_{i+1}^{n-1}. Keadaan ini dapat menyebabkan ketidakstabilan dari skema tersebut, yang berupa terjadinya amplifikasi hasil hitungan dari kondisi awal. Agar stabil dibutuhkan suatu syarat yaitu :

$$0 < \pi < ½ \text{ dengan } \pi = \frac{\Delta t}{\Delta x^2} k$$

Contoh:



Dimana : k = 1
Δx = 0,1
Δt = 0,001

$$\pi = \frac{\Delta t}{\Delta x^2} k = \frac{0{,}001}{0{,}1^2} = 0{,}1 < 0{,}5 \text{ (stabil)}$$

Syarat batas : pada t = 0 ; $T = 2x$; $0 \le x \le ½L$

$T = 2(1-x)$; $½L \le x \le L$

Dengan menggunakan persamaan (6.2), hitungan dilakukan dari i = 2 sampai dengan 5 dan dari n = 1 sampai waktu yang dikehendaki (N). Untuk n = 1 dan i bergerak dari i = 2 sampai i = 6,

$T_2^1 = 0{,}2 + 1 \,.\, 0{,}1 \,.\, (0 - 2 \,.\, 0{,}2 + 0{,}4) = 0{,}2$

$T_3^1 = 0{,}4 + 1 \,.\, 0{,}1 \,.\, (0{,}2 - 2 \,.\, 0{,}4 + 0{,}6) = 0{,}4$

$T_4^1 = 0{,}6 + 1 \,.\, 0{,}1 \,.\, (0{,}4 - 2 \,.\, 0{,}6 + 0{,}8) = 0{,}6$

$T_5^1 = 0{,}8 + 1 \,.\, 0{,}1 \,.\, (0{,}6 - 2 \,.\, 0{,}8 + 1) = 0{,}8$

$T_6^1 = 1 + 1 \,.\, 0{,}1 \,.\, (0{,}8 - 2 \,.\, 1 + 0{,}8) = 0{,}96$

untuk n = 2 dan i bergerak dari i =2 sampai i = 6,

$T_2^2 = 0{,}2 + 1 \,.\, 0{,}1 \,.\, (0 - 2 \,.\, 0{,}2 + 0{,}4) = 0{,}2$

$T_3^2 = 0{,}4 + 1 \,.\, 0{,}1 \,.\, (0{,}2 - 2 \,.\, 0{,}4 + 0{,}6) = 0{,}4$

$T_4^2 = 0{,}6 + 1 \,.\, 0{,}1 \,.\, (0{,}4 - 2 \,.\, 0{,}6 + 0{,}8) = 0{,}6$

$T_5^2 = 0{,}8 + 1 \,.\, 0{,}1 \,.\, (0{,}6 - 2 \,.\, 0{,}8 + 0{,}96) = 0{,}796$

$T_6^2 = 0{,}96 + 1 \,.\, 0{,}1 \,.\, (0{,}8 - 2 \,.\, 0{,}96 + 0{,}8) = 0{,}928$

Demikian perhitungan terus dilanjutkan s/d waktu yang dikehendaki (N).

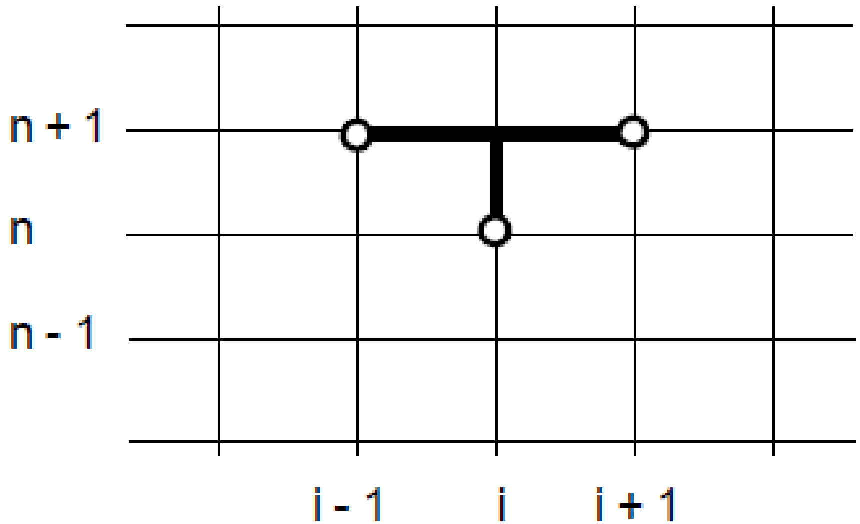
Tabel hasil skema eksplisit

i = x =	1 0	2 0.1	3 0.2	4 0.3	5 0.4	6 0.5	7 0.6
t = 0	**0**	**0.2**	**0.4**	0.6	0.8	1	0.8
t = 0,001	0	**0.2**	0.4	0.6	0.8	0.96	0.8
t = 0,002	0	0.2	0.4	0.6	0.796	0.928	0.796
t = 0,003	0	0.2	0.4	0.5996	0.7896	0.9016	0.7896
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
t = N	N	N	N	N	N	N	N

penyelesaian persamaan parabolik dengan skema implisit

Dalam skema eksplisit, ruas kanan dari persamaan ditulis pada waktu *n* yang nilainya sudah diketahui. Sedangkan pada skema implisit, ruas kanan tersebut ditulis pada waktu *n*+1 di mana nilainya belum diketahui.

Gambar di bawah ini menunjukkan jaringan titik simpul dari skema implisit. Dengan menggunakan skema tersebut, fungsi *f*(x,t) dan turunannya dalam ruang waktu didekati oleh bentuk berikut ini.



$$f(x,t) = f_i^n \text{ atau } = f_i^{n+1}$$

$$\frac{\partial f(x,t)}{\partial t} = \frac{f_i^{n+1} - f_i^n}{\Delta t}$$

$$\frac{\partial f(x,t)}{\partial x} = \frac{f_{i+1}^{n+1} - f_{i-1}^{n+1}}{2 \cdot \Delta x}$$

$$\frac{\partial^2 f(x,t)}{\partial x^2} = \frac{f_{i-1}^{n+1} - 2 \cdot f_i^{n+1} + f_{i+1}^{n+1}}{\Delta x^2}$$

Dengan menggunakan skema di atas, maka dapat dibentuk persamaan dalam bentuk beda hingga :

$$\frac{T_i^{n+1} - T_i^n}{?\, t} = K_i \, \frac{T_{i-1}^{n+1} - 2 \cdot T_i^{n+1} + T_{i+1}^{n+1}}{?\, x^2}$$

$$\frac{1}{?\, t} T_i^{n+1} - \frac{K_i}{?\, x^2} T_{i-1}^{n+1} + \frac{2 \cdot K_i}{?\, x^2} T_i^{n+1} - \frac{K_i}{?\, x^2} T_{i+1}^{n+1} = \frac{T_i^n}{?\, t}$$

$$- \frac{K_i}{?\, x^2} T_{i-1}^{n+1} + \left(\frac{1}{?\, t} + \frac{2 \cdot K_i}{?\, x^2} \right) T_i^{n+1} - \frac{K_i}{?\, x^2} T_{i+1}^{n+1} = \frac{T_i^n}{?\, t}$$

atau

$$A_i . T_{i-1}^{n+1} + B_i . T_i^{n+1} + C_i . T_{i+1}^{n+1} = D_i \quad .... \quad (6.3)$$

dengan

$$A_i = - \frac{K_i}{?\, x^2} \qquad ; C_i = - \frac{K_i}{?\, x^2}$$

$$B_i = \left(\frac{1}{?\, t} + 2 \cdot \frac{K_i}{?\, x^2} \right) \qquad ; D_i = \frac{T_i^n}{?\, t}$$

Apabila persamaan (6.3) ditulis untuk setiap titik hitungan dari i = 1 sampai M maka akan terbentuk suatu sisten persamaan linier yang dapat diselesaikan dengan menggunakan metode matriks.

Untuk :

$$i = 1 \quad ? \quad A_1 T_0 + B_1 T_1 + C_1 T_2 = D_1$$
$$i = 2 \quad ? \quad A_2 T_1 + B_2 T_2 + C_2 T_3 = D_2$$
$$i = 3 \quad ? \quad A_3 T_2 + B_3 T_3 + C_3 T_4 = D_3$$
$$i = 4 \quad ? \quad A_4 T_3 + B_4 T_4 + C_4 T_5 = D_4$$

.

.

$$i = M \quad ? \quad A_M T_{M-1} + B_M T_M + C_M T_{M+1} = D_M$$

Untuk penyederhanaan penulisan, variabel T_i^{n+1} ditulis T_i (tanpa menulis n+1). Persamaan di atas dalam bentuk matrik menjadi :

$$\begin{bmatrix} B_1 & C_1 & 0 & 0 & 0 & \ldots\ldots\ldots & 0 \\ A_2 & B_2 & C_2 & 0 & 0 & \ldots\ldots\ldots & 0 \\ 0 & A_3 & B_3 & C_3 & 0 & \ldots\ldots\ldots & 0 \\ 0 & 0 & A_4 & B_4 & C_4 & \ldots\ldots\ldots & 0 \\ . & . & . & . & . & \ldots\ldots\ldots & 0 \\ . & . & . & . & . & \ldots\ldots\ldots & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots\ldots A_M & B_M \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ . \\ . \\ T_M \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ . \\ . \\ D_M \end{bmatrix}$$