

# DERET DAN APROKSIMASI

DERET MACLAURIN  
DERET TAYLOR

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# TUJUAN

- Kenapa perlu perkiraan?
  - Perkiraan dibentuk dari fungsi paling sederhana – polynomial.
  - Kita bisa mengintegrasikan dan mendiferensiasi dengan mudah.
  - Kita bisa gunakan saat kita tidak tahu fungsi sebenarnya.

# POLYNOMIAL APPROXIMATIONS

- Misalkan kita ingin membuat perkiraan untuk sebuah fungsi yang kompleks pada **sekitar**  $x = 0$ ;
- Perkiraan paling simple adalah menentukan sebuah konstanta, sehingga:

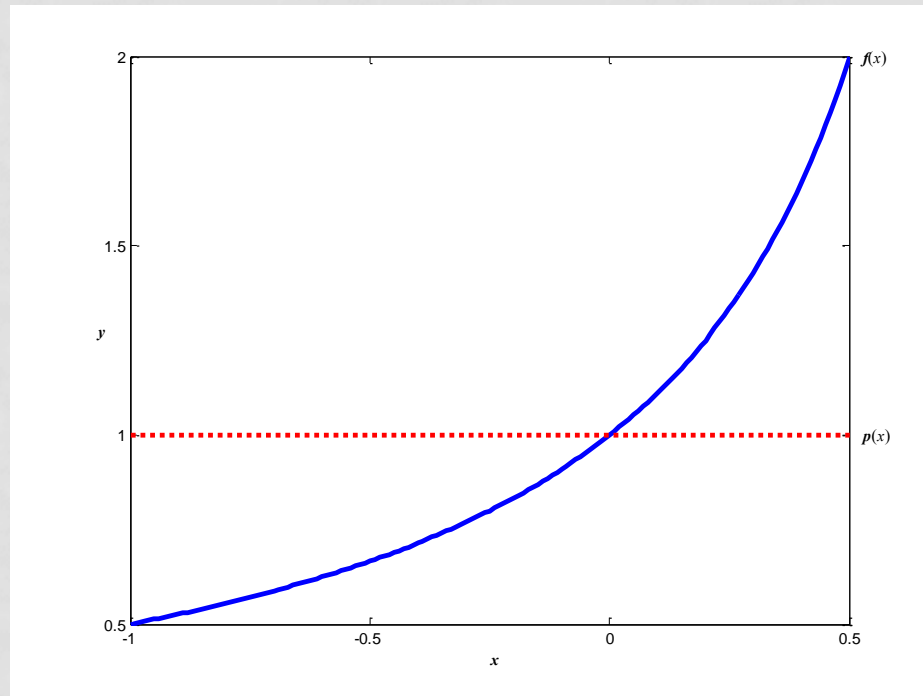
$$p_0(x) = a_0$$

- Catatan: perkiraan di atas disebut sebagai zero'th order polynomial approximation;
- Lalu, nilai berapa yang harus kita berikan pada konstanta itu?

# POLYNOMIAL APPROXIMATIONS

- Kita inginkan angka paling akurat pada  $x = 0$ .
- Sehingga:

$$p_0(x) = f(0)$$



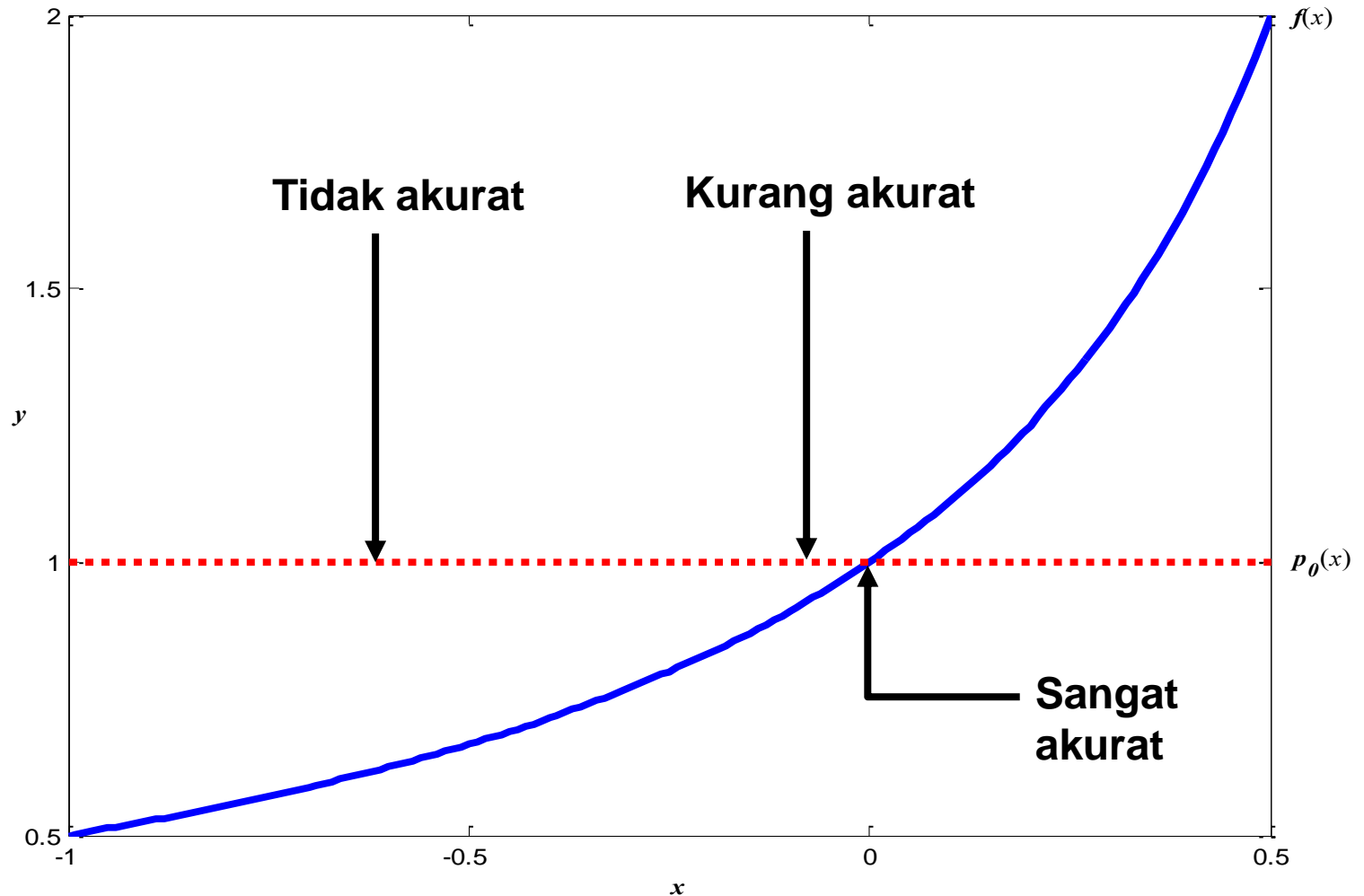
# POLYNOMIAL APPROXIMATIONS

- Contoh

$$f(x) = \frac{1}{1-x}$$

$$f(0) = \frac{1}{1} = 1 \Rightarrow p_0(x) = 1$$

# POLYNOMIAL APPROXIMATIONS



# POLYNOMIAL APPROXIMATIONS

- Sekarang kita tingkatkan dengan perkiraan dengan menggunakan aproksimasi linier (1<sup>st</sup> order approximation);

$$p_1(x) = a_0 + a_1x$$

- Sekarang kita pilih nilai sehingga perpotongan dan garis nya semirip mungkin dengan fungsi sebenarnya.

# POLYNOMIAL APPROXIMATIONS

- Menyamakan perpotongan:

$$\begin{aligned} p_1(0) = f(0) &\Rightarrow a_0 + a_1 \times 0 = f(0) \\ &\Rightarrow a_0 = f(0) \end{aligned}$$

- Menyamakan slope:

$$p_1'(0) = f'(0) \Rightarrow a_1 = f'(0)$$

- Sehingga polinomialnya:

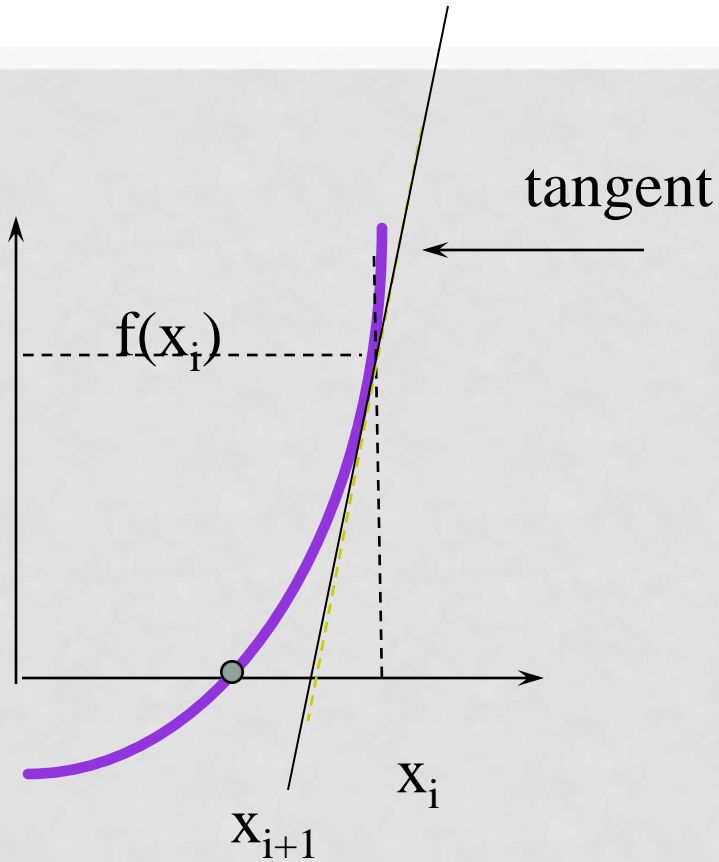
$$p_1(x) = f(0) + f'(0)x$$



# POLYNOMIAL APPROXIMATIONS

- **Contoh**  $f(x) = \frac{1}{1-x}$   
 $p_1(x) = a_0 + a_1x$   
 $f(0) = \frac{1}{1-0} = 1 \Rightarrow a_0 = f(0) = 1$   
 $f'(0) = \frac{1}{(1-x)^2} = 1 \Rightarrow a_1 = f'(0) = 1$   
 $\Rightarrow p_1(x) = 1 + x$

# INGAT, METODE NEWTON RAPHSON



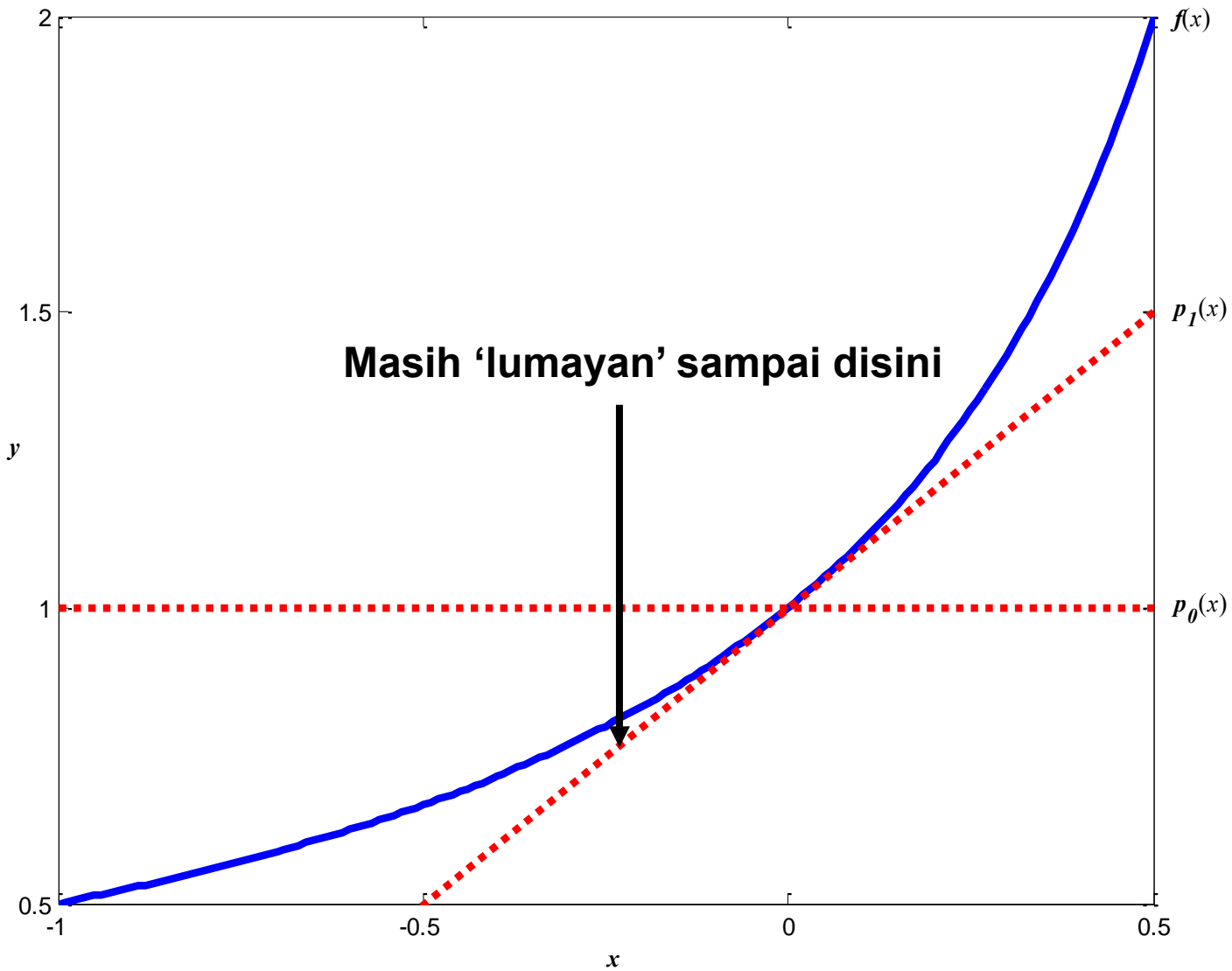
$$\text{tangent} = \frac{dy}{dx} = f'$$

$$f'(x_i) = \frac{f(x_i) - 0}{x_i - x_{i+1}}$$

*rearrange*

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

# DOT VARIANTS AT APPROXIMATIONS



# POLYNOMIAL APPROXIMATIONS

- Sekarang coba dengan perkiraan kuadrat:

$$p_2(x) = a_0 + a_1x + a_2x^2$$

- Kita inginkan perpotongan, gradient dan kurva (turunan kedua) dari perkiraan kita dapat *match* dengan fungsi sebenarnya pada  $x = 0$ .

# POLYNOMIAL APPROXIMATIONS

- Menyamakan perpotongan:

$$\begin{aligned} p_2(0) = f(0) &\Rightarrow a_0 + a_1 \times 0 + a_2 \times 0^2 = f(0) \\ &\Rightarrow a_0 = f(0) \end{aligned}$$

- Menyamakan kemiringan:

$$p_2'(0) = f'(0) \Rightarrow a_1 + 2a_2 \times 0 = f'(0)$$

# POLYNOMIAL APPROXIMATIONS

- Mencocokkan kurva (turunan ke 2):

$$p_2''(0) = f''(0) \quad \Rightarrow \quad 2a_2 = f''(0)$$

$$\Rightarrow a_2 = \frac{1}{2} f''(0)$$

- Memberikan polinom

$$p_2(x) = f(0) + f'(0)x + \frac{1}{2} f''(0)x^2$$

# POLYNOMIAL APPROXIMATIONS

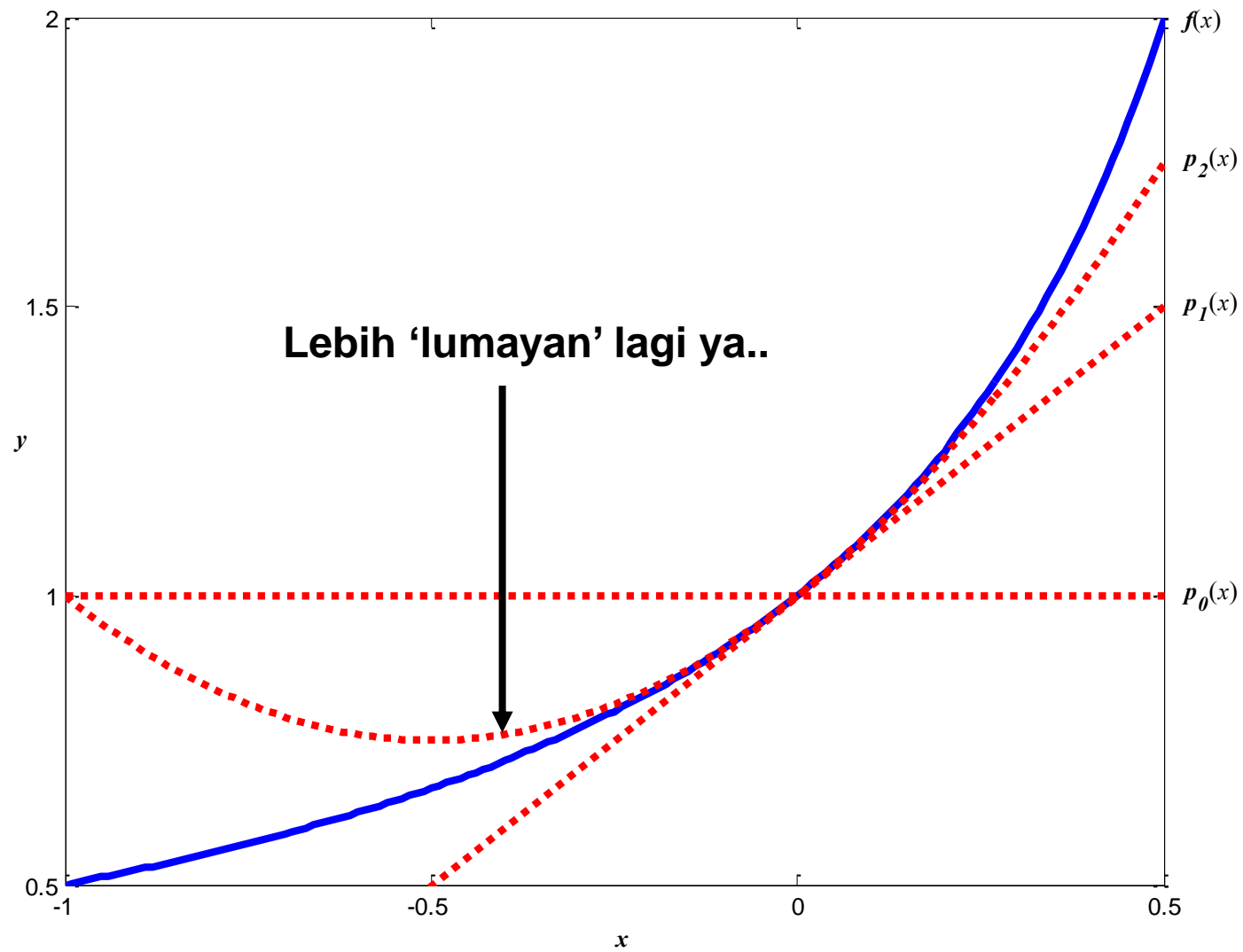
- Contoh  $f(x) = \frac{1}{1-x}$   
 $p_2(x) = a_0 + a_1x + a_2x^2$

- Dari sebelumnya:  $a_0 = 1, a_1 = 1$

$$f''(x) = \frac{2}{(1-x)^3} \Rightarrow 2a_2 = f''(0) = 2$$

$$\Rightarrow a_2 = 1$$

$$\Rightarrow p_2(x) = 1 + x + x^2$$





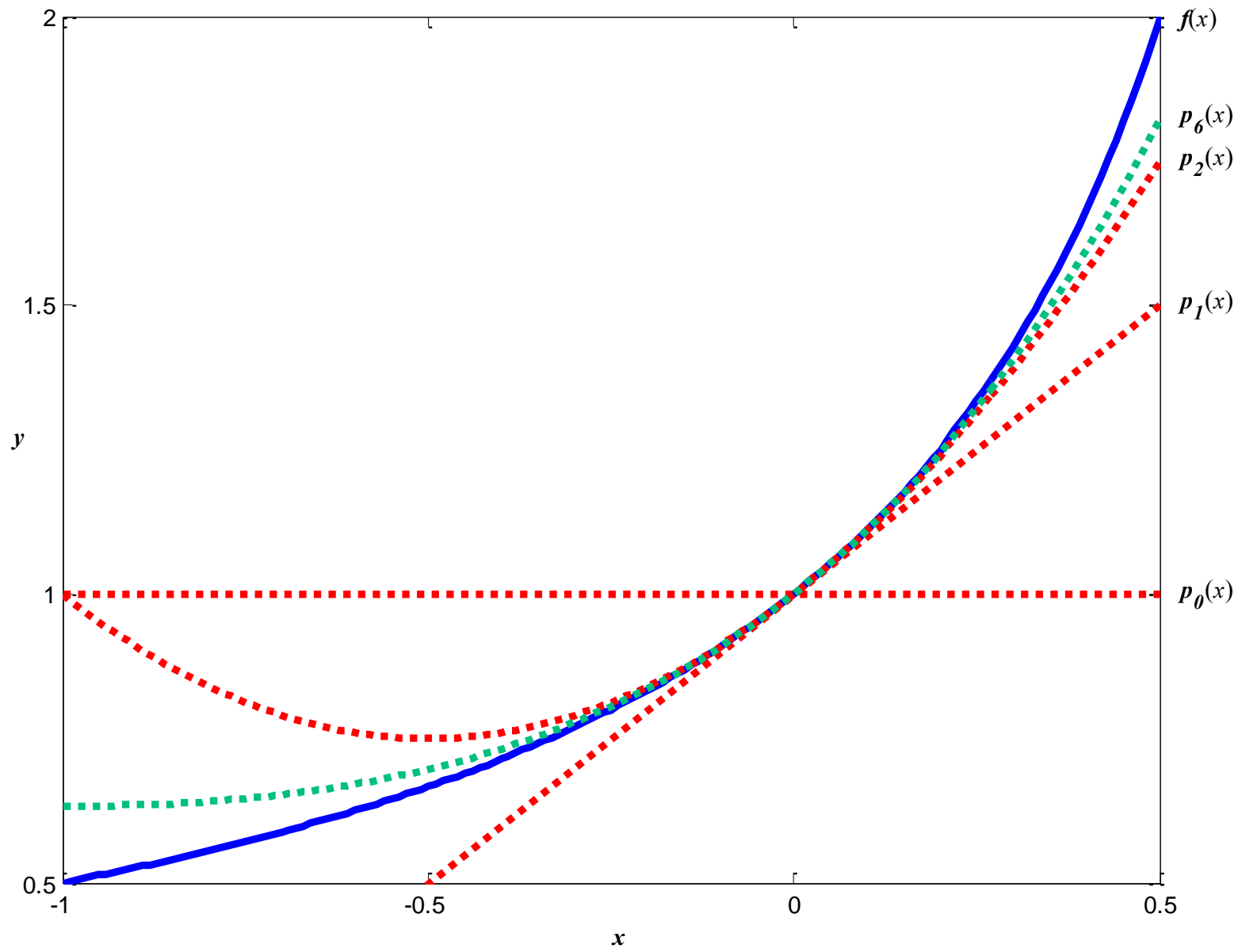
# POLYNOMIAL APPROXIMATIONS

- Kita bisa teruskan penaksiran secara polinom hingga  $n$  derajat.
- Kalau kita teruskan, kita akan mendapatkan rumus:

$$f(x) \approx p_n(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \dots + f^{(n)}(0)\frac{x^n}{n!}$$

# POLYNOMIAL APPROXIMATIONS

- Akurasi perkiraan akan bertambah seiring dengan penambahan polinom;
- Kita lihat polinom derajat 0, 1, 2 dan 6 (warna hijau), dibanding fungsi aslinya  $f(x)$  (warna biru).



# MACLAURIN (POWER) SERIES

- Deret Maclaurin adalah penaksiran polinom derajat tak hingga

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \dots + f^{(n)}(0)\frac{x^n}{n!} + \dots$$

- Notice: Deret infinite (tak hingga) menyatakan bahwa akhirnya deret ini sama dengan fungsi sebenarnya, bukan penaksiran lagi!

# TAYLOR SERIES

- Dari awal kita selalu memulai perkiraan pada nilai  $x = 0$
- Sesungguhnya, kita bisa membuat deret polinom yang berasal dari titik manapun.  $x = x_0$
- Ini disebut *Taylor Series*.
- **Jadi, Deret MacLaurin** merupakan Deret Taylor yang berpusat pada  $x_0=0$

# TAYLOR SERIES

- Rumus umum Deret Taylor:

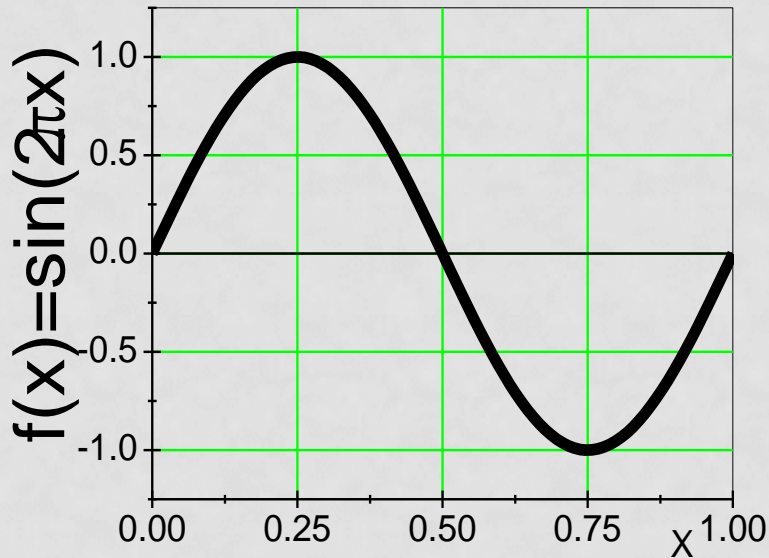
$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0) \frac{(x - x_0)^2}{2!}$$

$$+ \dots + f^{(n)}(x_0) \frac{(x - x_0)^n}{n!} + \dots$$

$$= \sum_{n=0}^{\infty} f^{(n)}(x_0) \frac{(x - x_0)^n}{n!}$$

# TAYLOR SERIES

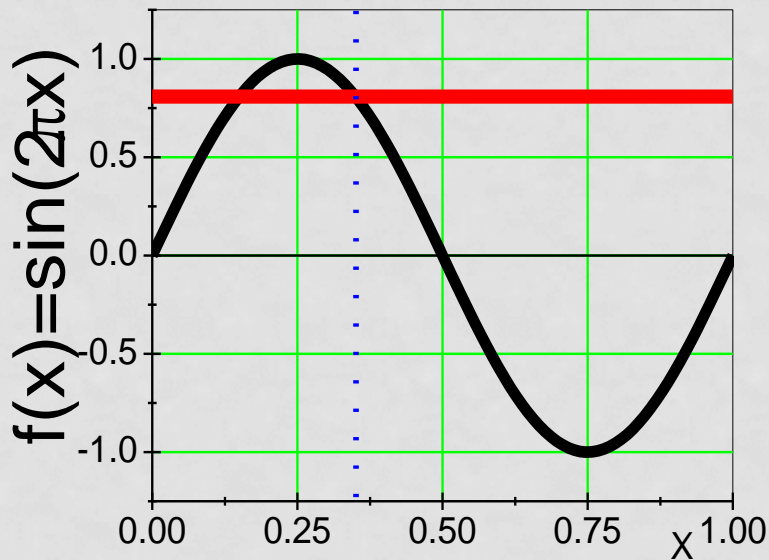
- Approximate function? Copy derivatives!



What is  $f(x)$  near  $x=0.35$ ?

# TAYLOR SERIES

- Approximate function? Copy derivatives!



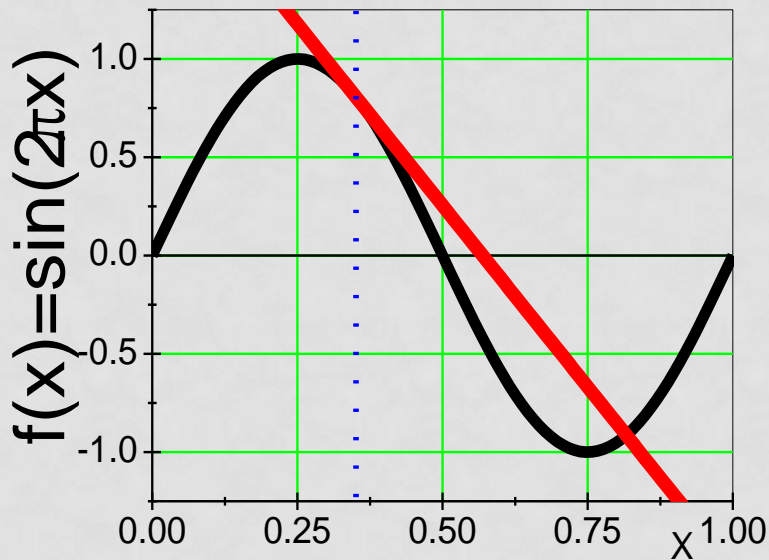
What is  $f(x)$  near  $x=0.35$ ?

$$T_0(x) = f(0.35)$$



# TAYLOR SERIES

- Approximate function? Copy derivatives!



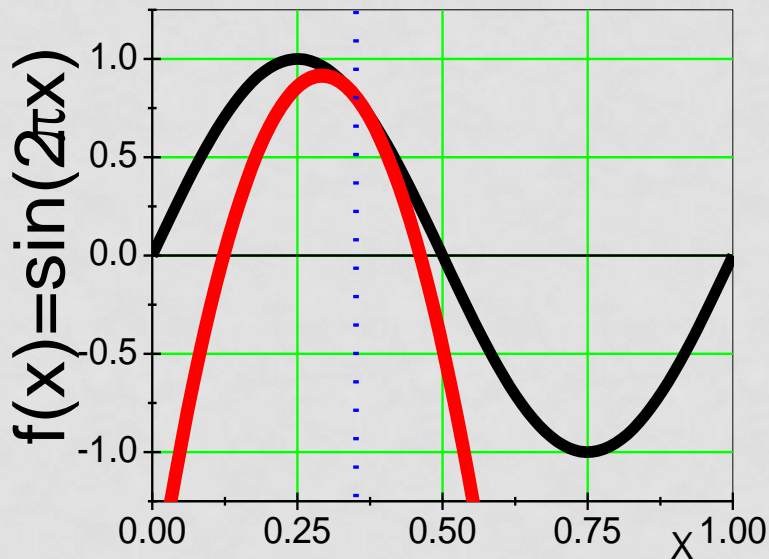
What is  $f(x)$  near  $x=0.35$ ?

$$T_1(x) = f(0.35)$$

$$+ f'(0.35)(x - 0.35)$$

# TAYLOR SERIES

- Approximate function? Copy derivatives!



What is  $f(x)$  near  $x=0.35$ ?

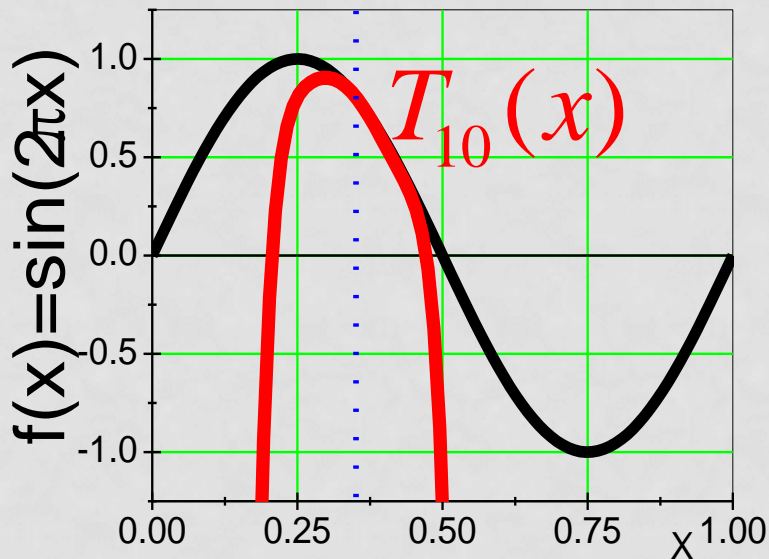
$$T_2(x) = f(0.35)$$

$$+ f'(0.35)(x - 0.35)$$

$$+ \frac{1}{2} f''(0.35)(x - 0.35)^2$$

# TAYLOR SERIES

- Approximate function? Copy derivatives!



What is  $f(x)$  near  $x=0.35$ ?

$$T_2(x) = f(0.35)$$

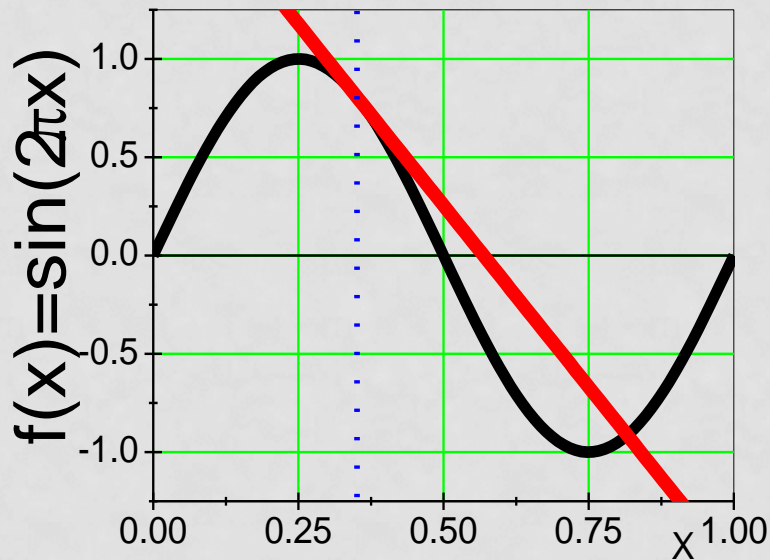
$$+ f'(0.35)(x - 0.35)$$

$$+ \frac{1}{2} f''(0.35)(x - 0.35)^2 \dots$$

$$T_N(x) = \sum_{i=0}^N \frac{f^{(i)}(a)(x-a)^i}{i!}$$

# TAYLOR SERIES

- Approximate function? Copy derivatives!



Most Common: 1<sup>st</sup> Order

$$T_1(x) = f(a) + f'(a)(x - a)$$

- Look out for “approximate” or “when  $x$  is small” or “small angle” or “close to” ...

# CONTOH – TAYLOR SERIES

- Bentuklah Deret Taylor untuk:

$$f(x) = \ln(x), \quad x_0 = 1$$

- Cari nilai fungsi dan turunannya untuk fungsi pada  $x_0=1$

# CONTOH – TAYLOR SERIES

$$f(x) = \ln(x) \quad \Rightarrow \quad f(x_0) = \ln(1) = 0$$

$$f'(x) = \frac{1}{x} \quad \Rightarrow \quad f'(x_0) = \frac{1}{1} = 1$$

$$f''(x) = -\frac{1}{x^2} \quad \Rightarrow \quad f''(x_0) = -\frac{1}{1^2} = -1$$

$$f'''(x) = \frac{2}{x^3} \quad \Rightarrow \quad f'''(x_0) = \frac{2}{1^3} = 2$$

⋮

$$f^{(n)}(x) = \frac{(n-1)!(-1)^{n-1}}{x^n}$$

$$\Rightarrow f^{(n)}(x_0) = \frac{(n-1)!(-1)^{n-1}}{1^n} = (n-1)!(-1)^{n-1}$$

# CONTOH – TAYLOR SERIES

- Gunakan Rumus Umum Deret Taylor:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0) \frac{(x - x_0)^2}{2!} \\ + \dots + f^{(n)}(x_0) \frac{(x - x_0)^n}{n!} + \dots$$

$$\Rightarrow \ln(x) = 0 + (x - 1) - \frac{(x - 1)^2}{2!} + \frac{2!(x - 1)^3}{3!} \\ + \dots + (n - 1)!(-1)^{n-1} \frac{(x - 1)^n}{n!} + \dots$$

$$\Rightarrow \ln(x) = (x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} \\ + \dots + (-1)^{n-1} \frac{(x - 1)^n}{n} + \dots$$

# TRUNCATED TAYLOR SERIES

- We cannot evaluate a Taylor series – it is infinite!
- Kita bisa memutuskan untuk membuat perkiraan dari sebuah fungsi hingga  $n$  (derajat) tertentu yang *tidak* tak terhingga;
- Kita sebut sebagai ***Truncated Taylor Series***.



# TRUNCATED TAYLOR SERIES

- To find an  $n$ th order truncated Taylor series

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + f''(x_0) \frac{(x - x_0)^2}{2!} \\ + \dots + f^{(n)}(x_0) \frac{(x - x_0)^n}{n!}$$

- **Note:** This is the same concept as the polynomial approximations we introduced earlier.

# EXAMPLE – TRUNCATED TAYLOR SERIES

- Find a cubic (degree 3) truncated Taylor series for the function:

$$f(x) = \cos(2x)$$

centered at:

$$x = \frac{\pi}{4}$$

# EXAMPLE – TRUNCATED TAYLOR SERIES

- For a degree 3 approximation:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + f''(x_0) \frac{(x - x_0)^2}{2!} + f'''(x_0) \frac{(x - x_0)^3}{3!}$$

- So we need to evaluate the function and its first three derivatives at the center.

# EXAMPLE – TRUNCATED TAYLOR SERIES

- Evaluating these:

$$f(x) = \cos(2x) \quad \Rightarrow \quad f\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

$$f'(x) = -2\sin(2x) \quad \Rightarrow \quad f'\left(\frac{\pi}{4}\right) = -2\sin\left(\frac{\pi}{2}\right) = -2$$

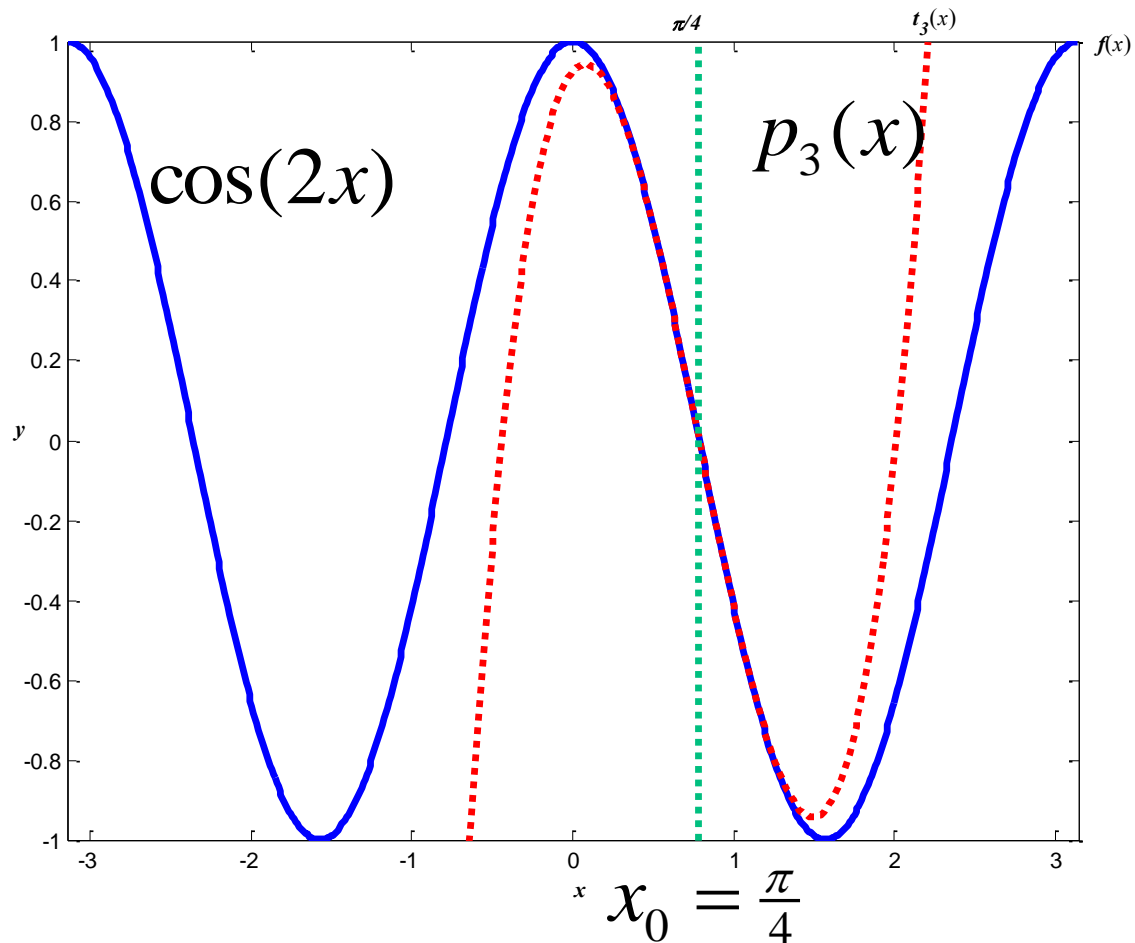
$$f''(x) = -4\cos(2x) \quad \Rightarrow \quad f''\left(\frac{\pi}{4}\right) = -4\cos\left(\frac{\pi}{2}\right) = 0$$

$$f'''(x) = 8\sin(2x) \quad \Rightarrow \quad f'''\left(\frac{\pi}{4}\right) = 8\sin\left(\frac{\pi}{2}\right) = 8$$

# EXAMPLE – TRUNCATED TAYLOR SERIES

- ... which gives:
$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + f''(x_0) \frac{(x - x_0)^2}{2!} + f'''(x_0) \frac{(x - x_0)^3}{3!}$$
$$\Rightarrow f(x) \approx 0 - 2 \times \left( x - \frac{\pi}{4} \right) + 0 \times \frac{\left( x - \frac{\pi}{4} \right)^2}{2!} + 8 \times \frac{\left( x - \frac{\pi}{4} \right)^3}{3!}$$
$$\Rightarrow f(x) \approx -2 \left( x - \frac{\pi}{4} \right) + \frac{4}{3} \left( x - \frac{\pi}{4} \right)^3$$

# EXAMPLE – TRUNCATED TAYLOR SERIES



# SERIES ACCURACY

- Kenapa mesti pakai Deret Taylor kalau bisa pakai Maclaurin?
- Perkiraan kita akan makin jauh dari akurat jika semakin jauh dari titik awal  $x_0$ ;
- Kita harus selalu memakai titik awal yang dekat dengan titik yang akan diperkirakan dan juga mudah untuk melakukan perkiraan.