SIMULTANEOUS LINEAR EQUATIONS

2013/9/9

GAUSS-SEIDEL METHOD

Adalah metode *ITERASI*

Prosedur dasar:

- Menyelesaikan tiap persamaan linier secara aljabar untuk x_i
- Membuat nilai asumsi solusi
- Selesaikan untuk tiap x_i dan ulangi
- Gunakan perkiraan kesalahan relatif tiap akhir iterasi untuk mengecek apakah error sudah mencapai angka toleransi.

GAUSS-SEIDEL METHOD

Kenapa?

Untuk mengatasi round-off error *(kesalahan pembulatan).*

Metode eliminasi seperti Gaussian Elimination and LU Decomposition(*) rawan terhadap kesalahan pembulatan.

GAUSS-SEIDEL METHOD

Algorithm

Sistem persamaan linier

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$\vdots \qquad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n$$

Kita mengubah sistem persamaan *[A]{X}={B}* untuk menyelesaikan *x1 dengan persamaan pertama*, menyelesaikan *x2 dengan persamaan kedua*, dan seterusnya.

GAUSS-SEIDEL METHOD

Algorithm

General Form of each equation

$$x_1 = \frac{c_1 - \sum_{\substack{j=1 \\ j\neq 1}}^{n} a_{1j} x_j}{a_{11}}$$

$$x_{n-1} = \frac{c_{n-1} - \sum_{\substack{j=1 \\ j\neq n-1}}^{n} a_{n-1,j} x_j}{a_{n-1,n-1}}$$

$$x_2 = \frac{c_2 - \sum_{\substack{j=1 \\ j\neq 2}}^{n} a_{2j} x_j}{a_{22}}$$

$$x_n = \frac{c_n - \sum_{\substack{j=1 \\ j\neq n}}^{n} a_{nj} x_j}{a_{nn}}$$

MENJADI:

Untuk sistem persamaan 3x3

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$

$$x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}}$$

$$x_3 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$$

Now we can start the solution process by choosing guesses for the x's. A simple way to obtain initial guesses is to assume that they are zero. These zeros can be substituted into $x_1$equation to calculate a new $x_1=b_1/a_{11}$.

First Iteration

$$x_1 = (c_1 - a_{12}x_2 - a_{13}x_3)/a_{11}$$
$$x_2 = (c_2 - a_{21}x_1 - a_{23}x_3)/a_{22}$$
$$x_3 = (c_3 - a_{31}x_1 - a_{32}x_2)/a_{33}$$

$$x_1 = (c_1 - a_{12}x_2 - a_{13}x_3)/a_{11}$$
$$x_2 = (c_2 - a_{21}x_1 - a_{23}x_3)/a_{22}$$
$$x_3 = (c_3 - a_{31}x_1 - a_{32}x_2)/a_{33}$$

Second Interation

$$x_1 = (c_1 - a_{12}x_2 - a_{13}x_3)/a_{11}$$
$$x_2 = (c_2 - a_{21}x_1 - a_{23}x_3)/a_{22}$$
$$x_3 = (c_3 - a_{31}x_1 - a_{32}x_2)/a_{33}$$

$$x_1 = (c_1 - a_{12}x_2 - a_{13}x_3)/a_{11}$$
$$x_2 = (c_2 - a_{21}x_1 - a_{23}x_3)/a_{22}$$
$$x_3 = (c_3 - a_{31}x_1 - a_{32}x_2)/a_{33}$$

(a) (b)

BATAS AKHIR ITERASI

New x_1 is substituted to calculate x_2 and x_3. The procedure is repeated until the convergence criterion is satisfied:

$$\left|\varepsilon_a\right|_i = \left|\frac{x_i^{baru} - x_i^{lama}}{x_i^{baru}}\right| \times 100 \prec \varepsilon_{diperkenankan}$$

$\left|\varepsilon_a\right| =$ ***a**pproximation error, sering digunakan, seringkali disebut sebagai galat **a**bsolut.*

$\left|\varepsilon_t\right| =$ True error, kurang berarti. → digunakan Relative error, dalam prosentase

GAUSS-SEIDEL METHOD: EXAMPLE 1

Diketahui sistem persamaan

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Initial Guess: asumsi nilai awal,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

GAUSS-SEIDEL METHOD: EXAMPLE 1

Tulis ulang untuk aplikasi Gauss-Seidel

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$x_1 = \frac{106.8 - 5x_2 - x_3}{25}$$

$$x_2 = \frac{177.2 - 64x_1 - x_3}{8}$$

$$x_3 = \frac{279.2 - 144x_1 - 12x_2}{1}$$

GAUSS-SEIDEL METHOD: EXAMPLE 1

Masukkan nilai perkiraan awal untuk selesaikan x_i

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

Initial Guess

$$x_1 = \frac{106.8 - 5(2) - (5)}{25} = 3.6720$$

$$x_2 = \frac{177.2 - 64(3.6720) - (5)}{8} = -7.8510$$

$$x_3 = \frac{279.2 - 144(3.6720) - 12(-7.8510)}{1} = -155.36$$

GAUSS-SEIDEL METHOD: EXAMPLE 1

Finding the absolute relative approximate error

$$\left|\varepsilon_a\right|_i = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| \times 100$$

$$\left|\varepsilon_a\right|_1 = \left|\frac{3.6720 - 1.0000}{3.6720}\right| x100 = 72.76\%$$

$$\left|\varepsilon_a\right|_2 = \left|\frac{-7.8510 - 2.0000}{-7.8510}\right| x100 = 125.47\%$$

$$\left|\varepsilon_a\right|_3 = \left|\frac{-155.36 - 5.0000}{-155.36}\right| x100 = 103.22\%$$

At the end of the first iteration

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3.6720 \\ -7.8510 \\ -155.36 \end{bmatrix}$$

The maximum absolute relative approximate error is 125.47%

GAUSS-SEIDEL METHOD: EXAMPLE 1

Iteration #2

Using

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3.6720 \\ -7.8510 \\ -155.36 \end{bmatrix}$$

from iteration #1

the values of a_i are found:

$$a_1 = \frac{106.8 - 5(-7.8510) - 155.36}{25} = 12.056$$

$$a_2 = \frac{177.2 - 64(12.056) - 155.36}{8} = -54.882$$

$$a_3 = \frac{279.2 - 144(12.056) - 12(-54.882)}{1} = -798.34$$

GAUSS-SEIDEL METHOD: EXAMPLE 1

Hitung “the absolute relative approximate error”

$$\left|\in_a\right|_1 = \left|\frac{12.056 - 3.6720}{12.056}\right| \text{x}100 = 69.542\%$$

$$\left|\in_a\right|_2 = \left|\frac{-54.882 - (-7.8510)}{-54.882}\right| \text{x}100 = 85.695\%$$

$$\left|\in_a\right|_3 = \left|\frac{-798.34 - (-155.36)}{-798.34}\right| \text{x}100 = 80.54\%$$

Akhir iterasi kedua

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 12.056 \\ -54.882 \\ -798.34 \end{bmatrix}$$

Galat absolut terbesar
85.695%

GAUSS-SEIDEL METHOD: EXAMPLE 1

Tersukan iterasi, kita dapatkan nilai berikut.

Iteration	a_1	$\left\|\in_a\right\|_1 \%$	a_2	$\left\|\in_a\right\|_2 \%$	a_3	$\left\|\in_a\right\|_3 \%$
1	3.672	72.767	-7.8510	125.47	-155.36	103.22
2	12.056	67.542	-54.882	85.695	-798.34	80.540
3	47.182	74.448	-255.51	78.521	-3448.9	76.852
4	193.33	75.595	-1093.4	76.632	-14440	76.116
5	800.53	75.850	-4577.2	76.112	-60072	75.962
6	3322.6	75.907	-19049	75.971	-249580	75.931

! Lho, kok? – Error nya nggak berkurang?

GAUSS-SEIDEL METHOD: PITFALL

Salahnya dimana?

Contoh tadi mengilustrasikan kemungkinan kesalahan pada Gauss-Siedel method: tidak semua sistem persamaan akan konvergen.

Is there a fix?

One class of system of equations always converges: One with a *diagonally dominant* coefficient matrix.

Diagonally dominant: [A] in [A] [X] = [C] is diagonally dominant if:

$$\left|a_{\mathrm{ii}}\right| \geq \sum_{\substack{j=1 \\ j \neq \mathrm{i}}}^{n} \left|a_{ij}\right|$$ Untuk semua ‘i’ ; DAN $$\left|a_{ii}\right\rangle \sum_{\substack{j=1 \\ j \neq i}}^{n} \left|a_{ij}\right|$$ Untuk minimal sebuah ‘i’

GAUSS-SEIDEL METHOD: PITFALL

Diagonally dominant: Koefisien pada diagonal harus sama atau lebih besar dari jumlah semua koefisien pada baris itu, dan minimal satu baris harus memiliki diagonal yang lebih besar dari jumlah koefisien pada baris itu.

Manakah matriks yang diagonally dominant?

$$[\mathrm{A}]=\begin{bmatrix} 2 & 5.81 & 34 \\ 45 & 43 & 1 \\ 123 & 16 & 1 \end{bmatrix}$$

$$[B]=\begin{bmatrix} 124 & 34 & 56 \\ 23 & 53 & 5 \\ 96 & 33 & 129 \end{bmatrix}$$

GAUSS-SEIDEL METHOD: EXAMPLE 2

Sistem persamaan linier

$$12x_1 + 3x_2 - 5x_3 = 1$$

$$x_1 + 5x_2 + 3x_3 = 28$$

$$3x_1 + 7x_2 + 13x_3 = 76$$

Dengan asumsi nilai awal

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Matriks Koefisien nya adalah

$$[A] = \begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$$

Akan konvergen kah?

GAUSS-SEIDEL METHOD: EXAMPLE 2

Cek apakah matriks nya diagonally dominant

$$[A]=\begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$$

$$|a_{11}| = |12| = 12 \geq |a_{12}| + |a_{13}| = |3| + |-5| = 8$$

$$|a_{22}| = |5| = 5 \geq |a_{21}| + |a_{23}| = |1| + |3| = 4$$

$$|a_{33}| = |13| = 13 \geq |a_{31}| + |a_{32}| = |3| + |7| = 10$$

Benar. Seharusnya konvergen dengan Gauss-Siedel Method

GAUSS-SEIDEL METHOD: EXAMPLE 2

Tulis ulang

$$\begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 28 \\ 76 \end{bmatrix}$$

$$x_1 = \frac{1 - 3x_2 + 5x_3}{12}$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$

$$x_3 = \frac{76 - 3x_1 - 7x_2}{13}$$

Asumsi nilai awal

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$x_1 = \frac{1 - 3(0) + 5(1)}{12} = 0.50000$$

$$x_2 = \frac{28 - (0.5) - 3(1)}{5} = 4.9000$$

$$x_3 = \frac{76 - 3(0.50000) - 7(4.9000)}{13} = 3.0923$$

GAUSS-SEIDEL METHOD: EXAMPLE 2

The absolute relative approximate error

$$\left|\in_a\right|_1 = \left|\frac{0.50000 - 1.0000}{0.50000}\right| \times 100 = 67.662\%$$

$$\left|\in_a\right|_2 = \left|\frac{4.9000 - 0}{4.9000}\right| \times 100 = 100.00\%$$

$$\left|\in_a\right|_3 = \left|\frac{3.0923 - 1.0000}{3.0923}\right| \times 100 = 67.662\%$$

Galat absolut terbesar di akhir iterasi pertama adalah 100%

GAUSS-SEIDEL METHOD: EXAMPLE 2

Setelah iterasi #1

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.5000 \\ 4.9000 \\ 3.0923 \end{bmatrix}$$

Masukkan nilai x pada persamaan

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.14679 \\ 3.7153 \\ 3.8118 \end{bmatrix}$$

$$x_1 = \frac{1 - 3(4.9000) + 5(3.0923)}{12} = 0.14679$$

$$x_2 = \frac{28 - (0.14679) - 3(3.0923)}{5} = 3.7153$$

$$x_3 = \frac{76 - 3(0.14679) - 7(3.7153)}{13} = 3.8118$$

Setelah iterasi #2

GAUSS-SEIDEL METHOD: EXAMPLE 2

Galat absolut dari Iterasi #2

$$\left|\in_a\right|_1 = \left|\frac{0.14679 - 0.50000}{0.14679}\right| \times 100 = 240.62\%$$

$$\left|\in_a\right|_2 = \left|\frac{3.7153 - 4.9000}{3.7153}\right| \times 100 = 31.887\%$$

$$\left|\in_a\right|_3 = \left|\frac{3.8118 - 3.0923}{3.8118}\right| \times 100 = 18.876\%$$

Galat absolut maksimum 240.62%

Lebih besar dari iterasi #1. Is this a problem?

GAUSS-SEIDEL METHOD: EXAMPLE 2

Ulangi iterasi, didapatkan…

Iteration	a_1	$\lvert\varepsilon_a\rvert_1$	a_2	$\lvert\varepsilon_a\rvert_2$	a_3	$\lvert\varepsilon_a\rvert_3$
1	0.50000	67.662	4.900	100.00	3.0923	67.662
2	0.14679	240.62	3.7153	31.887	3.8118	18.876
3	0.74275	80.23	3.1644	17.409	3.9708	4.0042
4	0.94675	21.547	3.0281	4.5012	3.9971	0.65798
5	0.99177	4.5394	3.0034	0.82240	4.0001	0.07499
6	0.99919	0.74260	3.0001	0.11000	4.0001	0.00000

Hasil akhir $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.99919 \\ 3.0001 \\ 4.0001 \end{bmatrix}$ Mendekati solusi sejati $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

LATIHAN

Sistem persamaan linier

$$3x_1 + 7x_2 + 13x_3 = 76$$

$$x_1 + 5x_2 + 3x_3 = 28$$

$$12x_1 + 3x_2 - 5x_3 = 1$$

With an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

GAUSS-SEIDEL METHOD

The Gauss-Seidel Method can still be used

The coefficient matrix is not diagonally dominant

$$[A]=\begin{bmatrix} 3 & 7 & 13 \\ 1 & 5 & 3 \\ 12 & 3 & -5 \end{bmatrix}$$

But this is the same set of equations used in example #2, which did converge.

$$[A]=\begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$$

If a system of linear equations is not diagonally dominant, check to see if rearranging the equations can form a diagonally dominant matrix.

GAUSS-SEIDEL METHOD

Not every system of equations can be rearranged to have a diagonally dominant coefficient matrix.

Observe the set of equations

$$x_1 + x_2 + x_3 = 3$$

$$2x_1 + 3x_2 + 4x_3 = 9$$

$$x_1 + 7x_2 + x_3 = 9$$

Which equation(s) prevents this set of equation from having a diagonally dominant coefficient matrix?

GAUSS-SEIDEL METHOD

Summary

-Advantages of the Gauss-Seidel Method

-Algorithm for the Gauss-Seidel Method

-Pitfalls of the Gauss-Seidel Method

GAUSS-SEIDEL METHOD

Questions?

METODE PENYELESAIAN

- Metode grafik
- Eliminasi Gauss
- Metode Gauss – Jourdan
- Metode Gauss – Seidel
- LU decomposition

LU DECOMPOSITION

$A=LU$

$Ax=b \Rightarrow LUx=b$

Define $Ux=y$

$Ly=b$ Solve y by forward substitution

$Ux=y$ Solve x by backward substitution

LU DECOMPOSITION BY GAUSSIAN ELIMINATION

There are infinitely many different ways to decompose A.
Most popular one: U=Gaussian eliminated matrix
L=Multipliers used for elimination

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ m_{2,1} & 1 & 0 & \cdots & 0 & 0 \\ m_{3,1} & m_{3,2} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & 0 \\ m_{n-1,1} & m_{n-1,2} & m_{n-1,3} & \cdots & 1 & \vdots \\ m_{n,1} & m_{n,2} & m_{n,3} & m_{n,4} & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} \\ 0 & 0 & a_{33}^{(3)} & \cdots & a_{3n}^{(3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & a_{n-1n-1}^{(n)} & a_{n-1n}^{(n)} \\ 0 & 0 & 0 & 0 & a_{nn}^{(n)} \end{bmatrix}$$

Compact storage: The diagonal entries of L matrix are all 1's, they don't need to be stored. LU is stored in a single matrix.

NEXT: SOLUSI PERSAMAAN NON LINIER

- Persamaan matematis yang sulit diselesaikan dengan “tangan” → analitis, sehingga diperlukan penyelesaian pendekatan → numerik
- Metode Numerik: Teknik menyelesaikan masalah matematika dengan pengoperasian hitungan, umumnya mencakup sejumlah besar kalkulasi aritmetika yang sangat banyak dan menjenuhkan
- Diselesaikan dengan algoritma (serangkaian perintah untuk menyelesaikan masalah), sehingga diperlukan bantuan komputer untuk melaksanakannya

SUMBER GALAT / ERROR

- Kesalahan pemodelan

  contoh: penggunaan hukum Newton

  asumsi benda adalah partikel
- Kesalahan bawaan

  contoh: kekeliruan dlm menyalin data

  salah membaca skala
- Ketidaktepatan data
- Kesalahan pemotongan / penyederhanaan persamaan(truncation error)
- Kesalahan pembulatan (round-off error)

SOLUSI PERSAMAAN NON LINEAR

1) Metode Akolade (bracketing method) / Closed method

- Metode Bagi dua (Bisection Method)
- Metode Regula Falsi (False Position Method)
- Metode Grafik

Kerugian: relatif lambat konvergen

Keuntungan: selalu konvergen

SOLUSI PERSAMAAN NON LINEAR

2) Metode Terbuka

Contoh:
- Iterasi Titik-Tetap (Fix Point Iteration)
- Metode Newton-Raphson
- Metode Secant

Keuntungan: cepat konvergen

Kerugian: tidak selalu konvergen