REVIEW OPERASI MATRIKS

TEKNIK LINGKUNGAN ITB

MENGHITUNG INVERS MATRIKS

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} c_{11}a_{11} + c_{12}a_{21} & \\ c_{11}a_{12} + c_{12}a_{22} & \\ & c_{21}a_{11} + c_{22}a_{21} \\ & c_{21}a_{12} + c_{22}a_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

DETERMINAN

→ Hanya untuk square matrices

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{vmatrix} a & b \\ c & d \end{vmatrix} \equiv ad - bc$$

$$\det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$\equiv a_1 b_2 c_3 - a_1 b_3 c_2 + a_2 b_3 c_1 - a_2 b_1 c_3 + a_3 b_1 c_2 - a_3 b_2 c_1$$

Jika determinan = 0 → matriks singular, tidak punya invers

CARI INVERS NYA…

$$\begin{bmatrix} 2 & 4 \\ 2 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

SISTEM PERSAMAAN LINEAR

METODE PENYELESAIAN

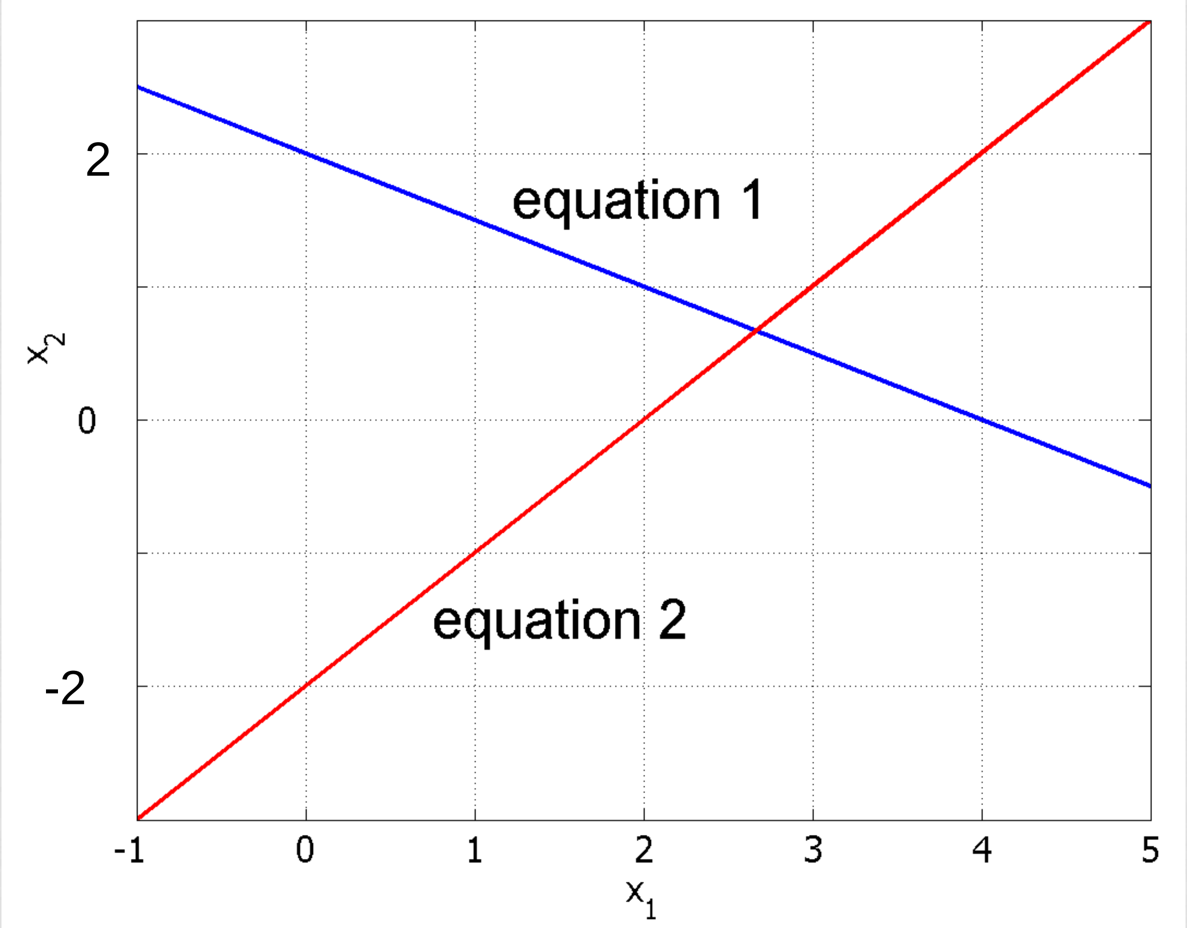
- Metode grafik
- Eliminasi Gauss
- Metode Gauss – Jourdan
- Metode Gauss – Seidel
- LU decomposition

METODE GRAFIK

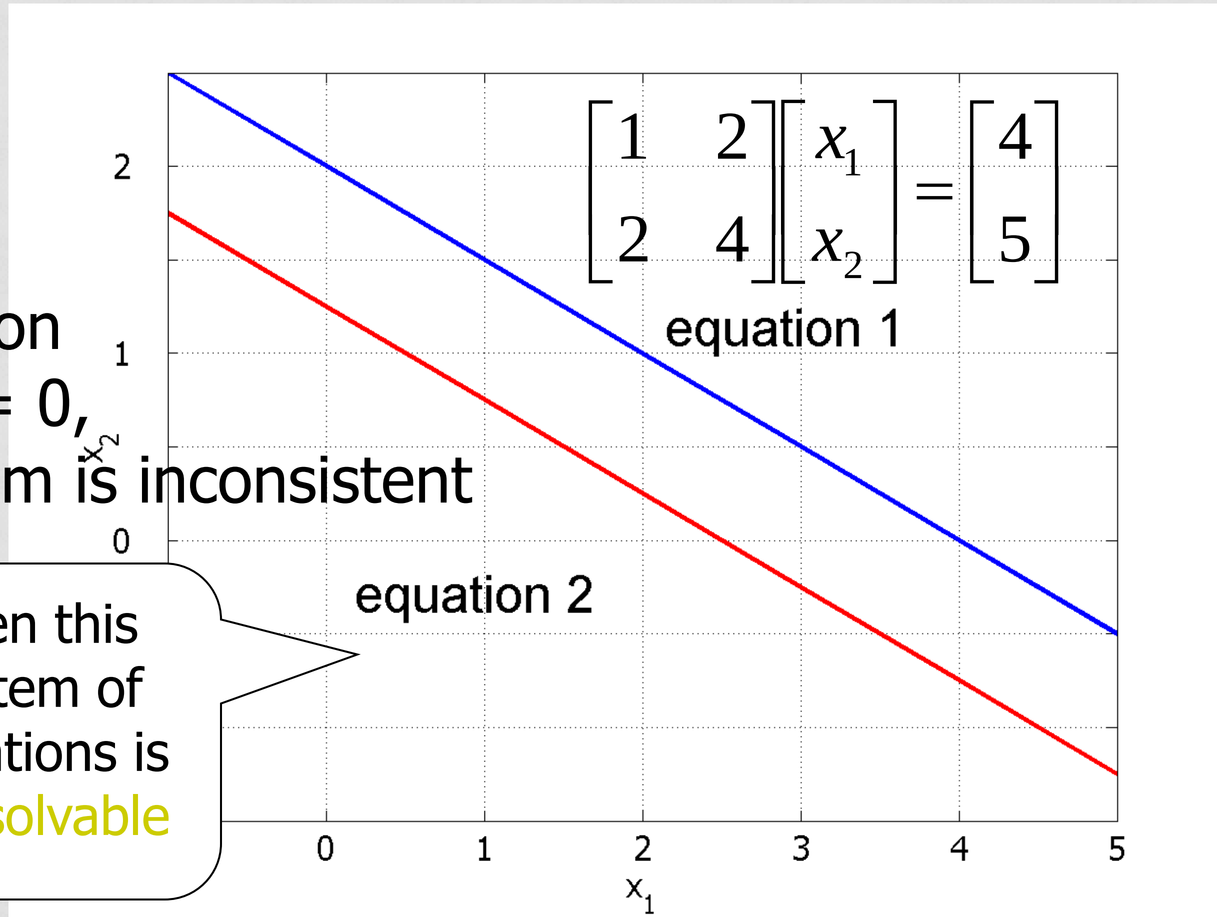
$$\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$\text{Det}\{A\} \neq 0 \Rightarrow A$ is nonsingular so invertible

→Unique solution



SISTEM PERSAMAAN YANG TAK TERSELESAIKAN



No solution
Det [A] = 0,
but system is inconsistent

Then this system of equations is not solvable

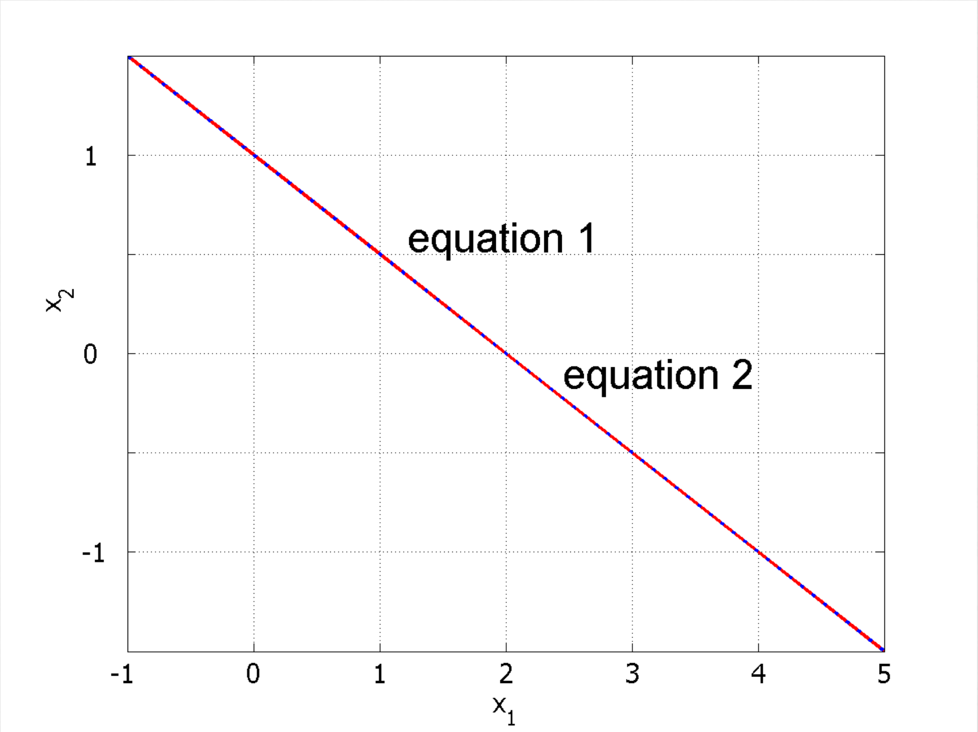
SISTEM DENGAN SOLUSI TAK TERBATAS

Det{A} = 0 $\Rightarrow$ A is singular

infinite number of solutions

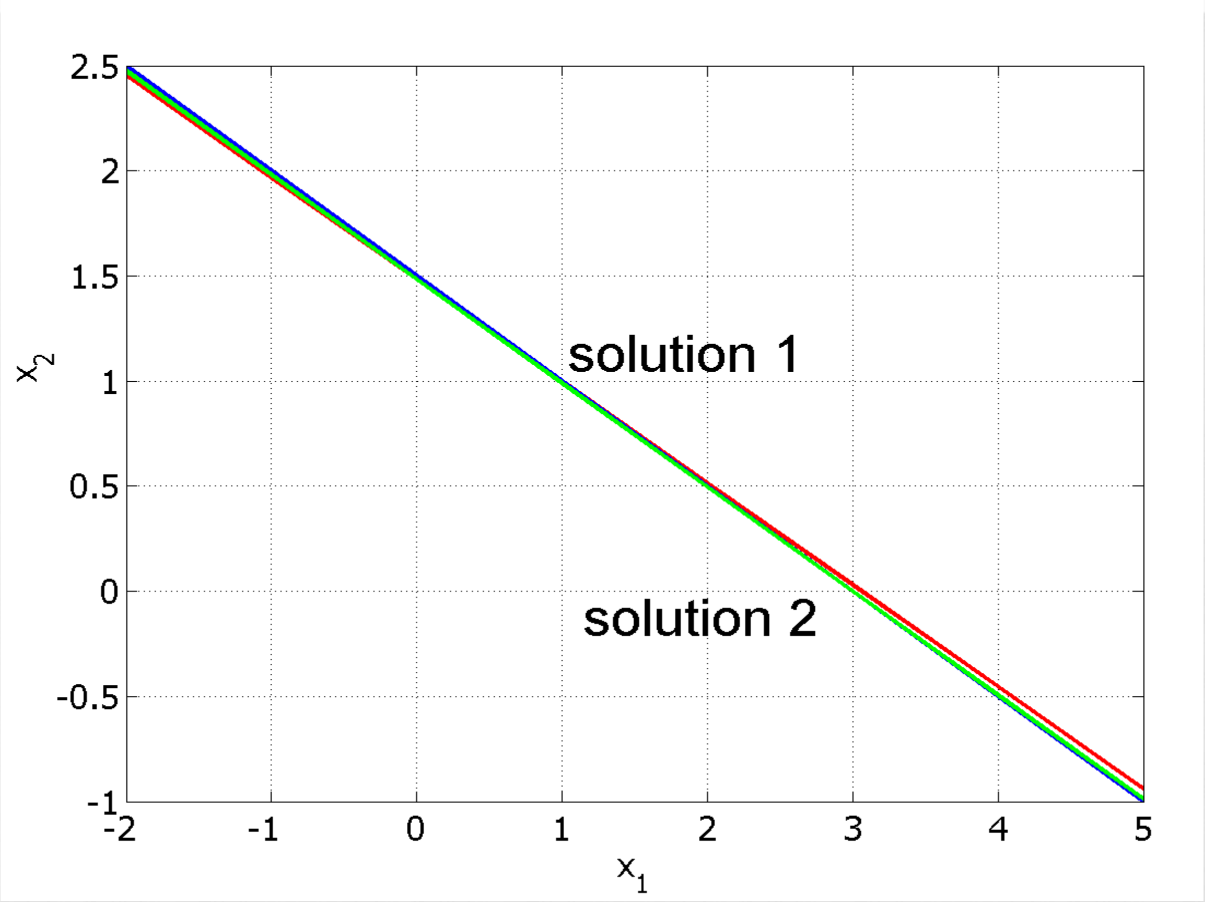
$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

Consistent so solvable



ILL-CONDITIONED SYSTEM OF EQUATIONS

A linear system of equations is said to be “ill-conditioned” if the coefficient matrix tends to be singular



ILL-CONDITIONED SYSTEM OF EQUATIONS

- A small deviation in the entries of A matrix, causes a large deviation in the solution.

$$\begin{bmatrix} 1 & 2 \\ 0.48 & 0.99 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1.47 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0.49 & 0.99 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1.47 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

GAUSSIAN ELIMINATION

Merupakan salah satu teknik paling populer dalam menyelesaikan sistem persamaan linear dalam bentuk:

$$[A][X]=[C]$$

Terdiri dari dua step

1. Forward Elimination of Unknowns.
2. Back Substitution

FORWARD ELIMINATION

Tujuan Forward Elimination adalah untuk membentuk matriks koefisien menjadi Upper Triangular Matrix

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

FORWARD ELIMINATION

Persamaan linear

n persamaan dengan *n* variabel yang tak diketahui

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$\begin{matrix} . & & . \\ . & & . \\ . & & . \end{matrix}$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n$$

CONTOH

$$2x_1 + 3x_2 - 2x_3 - x_4 = -2$$

$$2x_1 + 5x_2 - 3x_3 + x_4 = 7$$

$$-2x_1 + x_2 + 3x_3 - 2x_4 = 1$$

$$-5x_1 + 2x_2 - x_3 + 3x_4 = 8$$

matriks input

$$\begin{bmatrix} 2 & 3 & -2 & -1 & -2 \\ 2 & 5 & -3 & 1 & 7 \\ -2 & 1 & 3 & -2 & 1 \\ -5 & 2 & 1 & 3 & 8 \end{bmatrix}$$

FORWARD ELIMINATION

$$\begin{bmatrix} 2 & 3 & -2 & -1 & -2 \\ 2 & 5 & -3 & 1 & 7 \\ -2 & 1 & 3 & -2 & 1 \\ -5 & 2 & 1 & 3 & 8 \end{bmatrix}$$

$$R_1' = R_1/2$$
$$R_2' = R_2 - 2R_1'$$
$$R_3' = R_3 + 2R_1'$$
$$R_4' = R_4 + 5R_1'$$

$$\begin{bmatrix} 1 & 3/2 & -1 & -1/2 & -1 \\ 0 & 2 & -1 & 2 & 9 \\ 0 & 4 & 1 & -3 & -1 \\ 0 & 19/2 & -6 & 1/2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3/2 & -1 & -1/2 & -1 \\ 0 & 2 & -1 & 2 & 9 \\ 0 & 4 & 1 & -3 & -1 \\ 0 & 19/2 & -6 & 1/2 & 3 \end{bmatrix}$$

$$R_1' = R_1$$
$$R_2' = R_2/2$$
$$R_3' = R_3 + 4R_2'$$
$$R_4' = R_4 + 19/2\, R_1'$$

$$\begin{bmatrix} 1 & 3/2 & -1 & -1/2 & -1 \\ 0 & 1 & -1/2 & 1 & 9/2 \\ 0 & 0 & 3 & -7 & -19 \\ 0 & 0 & -5/4 & -9 & -159/4 \end{bmatrix}$$

FORWARD ELIMINATION

$$\begin{bmatrix} 1 & 3/2 & -1 & -1/2 & -1 \\ 0 & 1 & -1/2 & 1 & 9/2 \\ 0 & 0 & 3 & -7 & -19 \\ 0 & 0 & -5/4 & -9 & -159/4 \end{bmatrix}$$

$$R_1' = R_1$$
$$R_2' = R_2$$
$$R_3' = R_3/3$$
$$R_4' = R_4 + 5/4\, R_3'$$

$$\begin{bmatrix} 1 & 3/2 & -1 & -1/2 & -1 \\ 0 & 1 & -1/2 & 1 & 9/2 \\ 0 & 0 & 1 & -7/3 & -19/3 \\ 0 & 0 & 0 & -143/12 & -572/12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3/2 & -1 & -1/2 & -1 \\ 0 & 1 & -1/2 & 1 & 9/2 \\ 0 & 0 & 1 & -7/3 & -19/3 \\ 0 & 0 & 0 & -143/12 & -572/12 \end{bmatrix}$$

$$R_1' = R_1$$
$$R_2' = R_2$$
$$R_3' = R_3$$
$$R_4' = R_4 / (-143/12)$$

$$\begin{bmatrix} 1 & 3/2 & -1 & -1/2 & -1 \\ 0 & 1 & -1/2 & 1 & 9/2 \\ 0 & 0 & 1 & -7/3 & -19/3 \\ 0 & 0 & 0 & 1 & 572/143 \end{bmatrix}$$

BACK SUBSTITUTION

$$
\begin{aligned}
x_1 + \tfrac{3}{2}x_2 - x_3 - \tfrac{1}{2}x_4 &= -1 \\
x_2 - \tfrac{1}{2}x_3 + x_4 &= \tfrac{9}{2} \\
x_3 - \tfrac{7}{3}x_4 &= -\tfrac{19}{3} \\
x_4 &= \tfrac{572}{143}
\end{aligned}
$$

$$x_4 = 4$$

GAUSS - JOURDAN

$$\begin{bmatrix} 1 & 3 & 2 & 15 \\ 2 & 4 & 3 & 22 \\ 3 & 4 & 7 & 39 \end{bmatrix} \rightarrow \begin{aligned} R_1' &= R_1 \\ R_2' &= R_2 - 2R_1' \\ R_3' &= R_3 - 3R_1' \end{aligned} \downarrow \begin{bmatrix} 1 & 3 & 2 & 15 \\ 0 & -2 & -1 & -8 \\ 0 & -5 & 1 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 2 & 15 \\ 0 & -2 & -1 & -8 \\ 0 & -5 & 1 & -6 \end{bmatrix} \rightarrow \begin{aligned} R_1' &= R_1 - 3R_2' \\ R_2' &= -R_2/2 \\ R_3' &= R_3 + 5R_2' \end{aligned} \rightarrow \begin{bmatrix} 1 & 0 & 1/2 & 15 \\ 0 & 1 & 1/2 & 4 \\ 0 & 0 & 7/2 & 14 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1/2 & 15 \\ 0 & 1 & 1/2 & 4 \\ 0 & 0 & 7/2 & 14 \end{bmatrix} \rightarrow \begin{aligned} R_3' &= R_3 / (7/2) \\ R_1' &= R_1 - \tfrac{1}{2} R_3' \\ R_2' &= R_2 - \tfrac{1}{2} R_3' \end{aligned} \uparrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

WARNING..

Dua kemungkinan kesalahan

-Pembagian dengan nol mungkin terjadi pada langkah forward elimination. Misalkan:

$$10x_1 - 7x_2 = 7$$

$$6x_3 + 2.099x_2 - 3x_1 = 3.901$$

$$5x_1 - x_2 + 5x_3 = 6$$

- Kemungkinan error karena round-off (**kesalahan pembulatan)**

CONTOH

Dari sistem persamaan linear

$$\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 10 & -7 & 0 & 7 \\ -3 & 2.099 & 6 & 3.901 \\ 5 & -1 & 5 & 6 \end{bmatrix}$$

Akhir dari Forward Elimination

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 0 & 15005 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 15004 \end{bmatrix} \longrightarrow \begin{bmatrix} 10 & -7 & 0 & 7 \\ 0 & -0.001 & 6 & 6.001 \\ 0 & 0 & 15005 & 15004 \end{bmatrix}$$

KESALAHAN YANG MUNGKIN TERJADI

Back Substitution

$$x_3 = \frac{15004}{15005} = 0.99993$$

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 0 & 15005 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 15004 \end{bmatrix}$$

$$x_2 = \frac{6.001 - 6x_3}{-0.001} = -1.5$$

$$x_1 = \frac{7 + 7x_2 - 0x_3}{10} = -0.3500$$

CONTOH KESALAHAN

Bandung-kan solusi exact dengan hasil perhitungan

$$[X]_{exact} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$[X]_{calculated} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -0.35 \\ -1.5 \\ 0.99993 \end{bmatrix}$$

IMPROVEMENTS

Menambah jumlah angka penting

- Mengurangi round-off error **(kesalahan pembulatan)**
- Tidak menghindarkan pembagian dengan nol

Gaussian Elimination with Partial Pivoting

- **Menghindarkan pembagian dengan nol**
- **Mengurangi round-off error**

PIVOTING

Eliminasi Gauss dengan partial pivoting mengubah tata urutan baris untuk bisa mengaplikasikan Eliminasi Gauss secara Normal

How?

Di awal sebelum langkah ke-k pada forward elimination, temukan angka maksimum dari:

$$|a_{pk}| \quad |a_{kk}|, |a_{k+1,k}|, \ldots\ldots\ldots\ldots\ldots\ldots, |a_{nk}| \quad k \leq p \leq n,$$

Jika nilai maksimumnya Pada baris ke p,

Maka tukar baris p dan k.

PARTIAL PIVOTING

What does it Mean?

Gaussian Elimination with Partial Pivoting ensures that each step of Forward Elimination is performed with the pivoting element $|a_{kk}|$ having the largest absolute value.

Jadi,

Kita mengecek pada setiap langkah apakah angka mutlak yang dipakai untuk forward elimination (pivoting element) adalah selalu paling besar

PARTIAL PIVOTING: EXAMPLE

Consider the system of equations

$$10x_1 - 7x_2 = 7$$
$$-3x_1 + 2.099x_2 + 6x_3 = 3.901$$
$$5x_1 - x_2 + 5x_3 = 6$$

In matrix form

$$\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix}$$

Solve using Gaussian Elimination with Partial Pivoting using five significant digits with chopping

PARTIAL PIVOTING: EXAMPLE

Forward Elimination: Step 1

Examining the values of the first column

|10|, |-3|, and |5| or 10, 3, and 5

The largest absolute value is 10, which means, to follow the rules of Partial Pivoting, we don't need to switch the rows

Performing Forward Elimination

$$\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix}$$

PARTIAL PIVOTING: EXAMPLE

Forward Elimination: Step 2

Examining the values of the first column

|-0.001| and |2.5| or 0.0001 and 2.5

The largest absolute value is 2.5, so row 2 is switched with row 3

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & -0.001 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.001 \end{bmatrix}$$

Performing the row swap

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix} \Rightarrow$$

PARTIAL PIVOTING: EXAMPLE

Forward Elimination: Step 2

Performing the Forward Elimination results in:

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.002 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.002 \end{bmatrix}$$

PARTIAL PIVOTING: EXAMPLE

Back Substitution

Solving the equations through back substitution

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.002 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.002 \end{bmatrix}$$

$$x_3 = \frac{6.002}{6.002} = 1$$

$$x_2 = \frac{2.5 - 5x_2}{2.5} = -1$$

$$x_1 = \frac{7 + 7x_2 - 0x_3}{10} = 0$$

PARTIAL PIVOTING: EXAMPLE

Compare the calculated and exact solution

The fact that they are equal is coincidence, but it does illustrate the advantage of Partial Pivoting

$$[X]_{calculated} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \qquad [X]_{exact} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

SUMMARY

-Forward Elimination

-Back Substitution

-Pitfalls

-Improvements

-Partial Pivoting