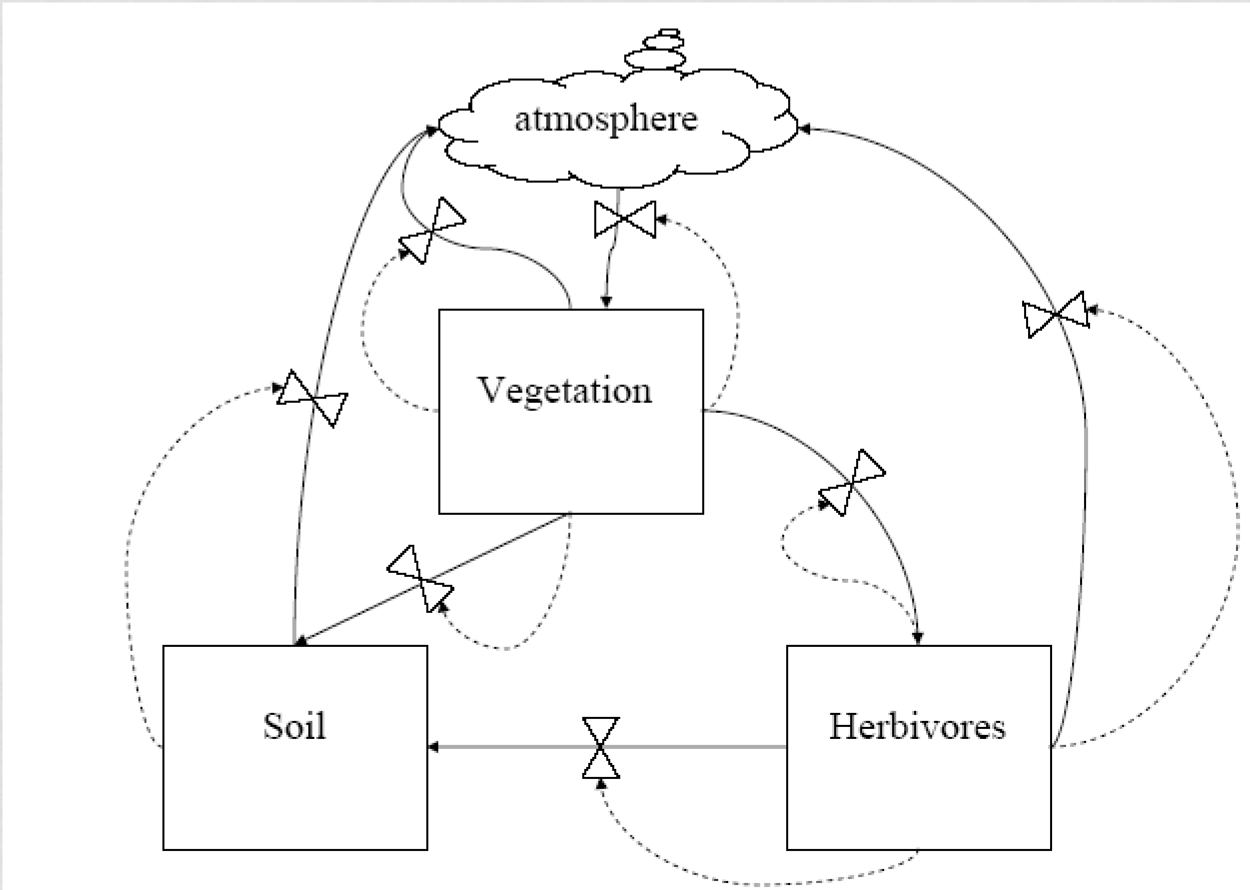
PERSAMAAN DIFERENSIAL
(DIFFERENTIAL EQUATION)

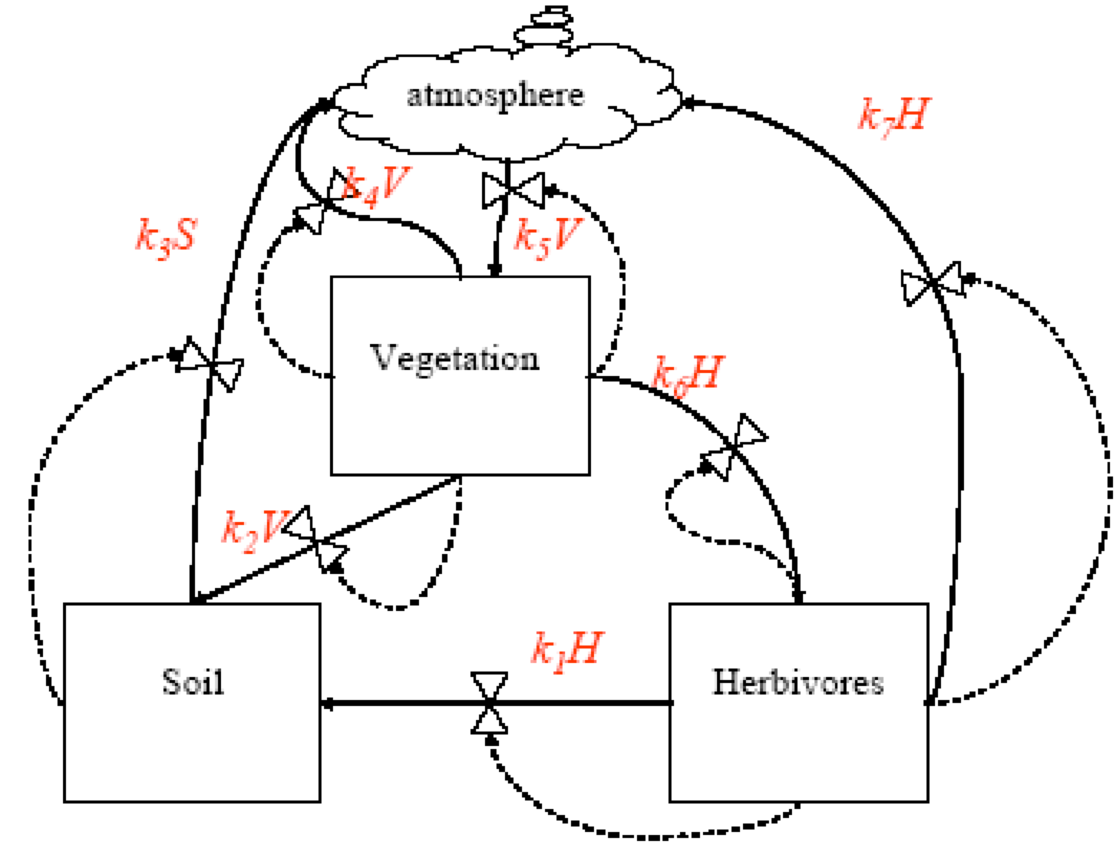
METODE EULER
METODE RUNGE-KUTTA

PERSAMAAN DIFERENSIAL

- Persamaan paling penting dalam bidang rekayasa, paling bisa menjelaskan apa yang terjadi dalam sistem fisik.
- Menghitung jarak terhadap waktu dengan kecepatan tertentu, 50 misalnya.

$$\frac{dx}{dt} = 50$$


atmosphere
Vegetation
Soil
Herbivores



$$\frac{dS}{dt} = k_1H + k_2V - k_3S$$

$$\frac{dV}{dt} = k_5V - k_2V - k_4V - k_6H$$

$$\frac{dH}{dt} = k_6H - k_1H - k_7H$$

} Rate equations

PERSAMAAN DIFERENSIAL

- Solusinya, secara analitik dengan integral,

$$\int dx = \int 50dt \longrightarrow x = 50t + C$$

- C adalah konstanta integrasi
- Artinya, solusi analitis tersebut terdiri dari banyak ‘alternatif’
- C hanya bisa dicari jika mengetahui nilai x dan t. Sehingga, untuk contoh di atas, jika x(0) = (x saat t=0) = 0, maka C = 0

KLASIFIKASI PERSAMAAN DIFERENSIAL

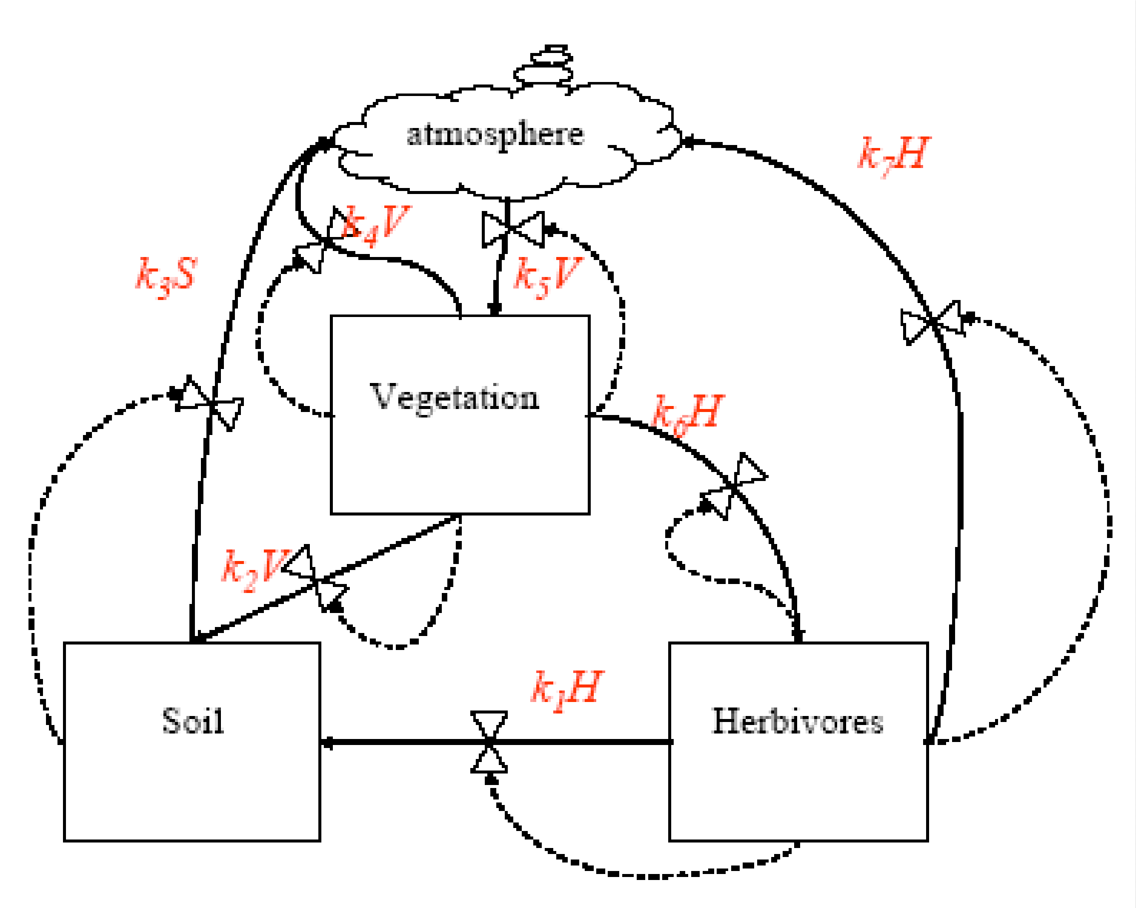
Persamaan yang mengandung turunan dari satu atau lebih variabel tak bebas, terhadap satu atau lebih variabel bebas.

- Dibedakan menurut:
  - Tipe (ordiner/biasa atau parsial)
  - Orde (ditentukan oleh turunan tertinggi yang ada)
  - Liniarity (linier atau non-linier)

SOLUSI PERSAMAAN DIFERENSIAL

- Secara analitik, mencari solusi persamaan diferensial adalah dengan mencari fungsi integral nya.
- Contoh, untuk fungsi pertumbuhan secara eksponensial, persamaan umum:

$$\frac{dP}{dt} = kP$$



$$\frac{dS}{dt} = k_1 H + k_2 V - k_3 S$$

$$\frac{dV}{dt} = k_5 V - k_2 V - k_4 V - k_6 H$$

$$\frac{dH}{dt} = k_6 H - k_1 H - k_7 H$$

Rate equations

But what you really want to know is…

the sizes of the boxes (or state variables) and how they change through time

That is, you want to know:

the state equations

There are two basic ways of finding the state equations for the state variables based on your known rate equations:

1) Analytical integration
2) Numerical integration

Suatu kultur bakteria tumbuh dengan kecepatan yang proporsional dengan jumlah bakteria yang ada pada setiap waktu. Diketahui bahwa jumlah bakteri bertambah menjadi dua kali lipat setiap 5 jam. Jika kultur tersebut berjumlah satu unit pada saat $t = 0$, berapa kira-kira jumlah bakteri setelah satu jam?

SOLUSI PERSAMAAN DIFERENSIAL

- Jumlah bakteri menjadi dua kali lipat setiap 5 jam, maka k = (ln 2)/5
- Jika P_0 = 1 unit, maka setelah satu jam…

$$\frac{dP}{dt} = kP$$

$$\int_{P_0}^{P_1} \frac{dP}{P} = \int_{t_0}^{t_1} k\,dt$$

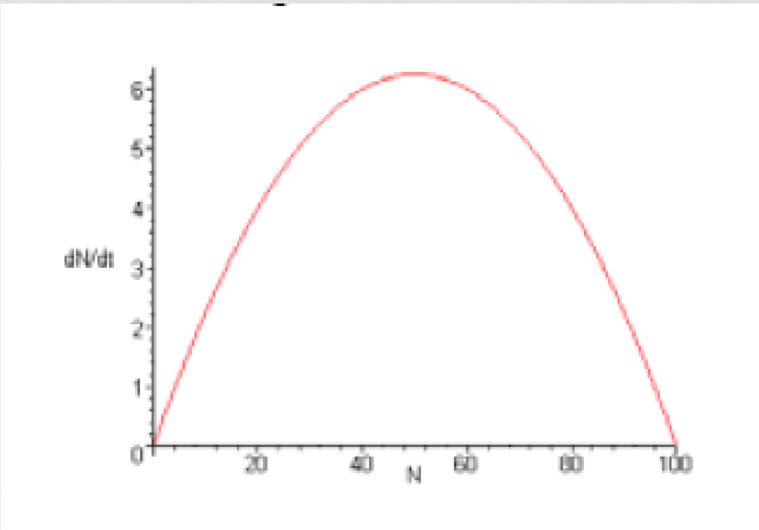
$$\ln \frac{P}{P_0} = Ck(t - t_0)$$

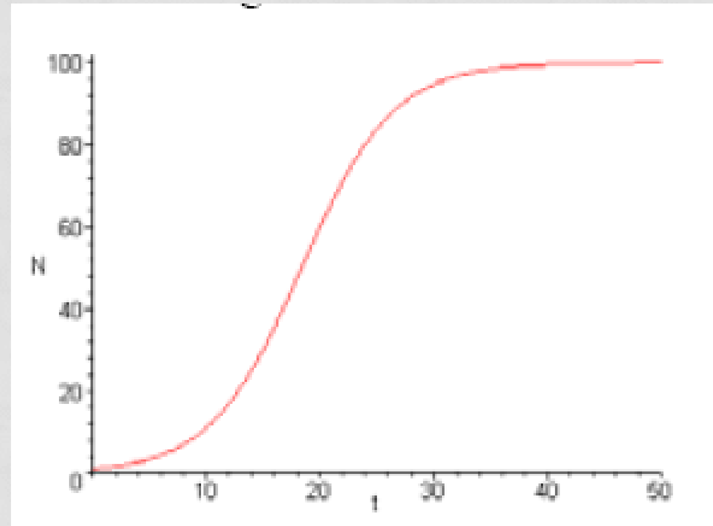
$$P(t) = P_0 e^{kt}$$

$$P(1) = 1\left(e^{\left(\frac{(\ln 2)}{5}\right)(1)}\right)$$

$$= 1.1487$$

The Analytical Solution of the Rate Equation is the State Equation





$$G(N) = \frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \xrightarrow[\text{“integration”}]{\text{solve by}} g(t) = N = \frac{K}{1 + \left(\frac{K - N_0}{N_0}\right)e^{-rt}}$$

Rate equation (dsolve in Maple) State equation

THERE ARE VERY FEW MODELS IN ECOLOGY THAT CAN BE SOLVED ANALYTICALLY.

SOLUSI NUMERIK

- Numerical integration
  - Eulers
  - Runge-Kutta

Numerical integration makes use of this relationship:

$$y_{t+\Delta t} \approx y_t + \frac{dy}{dt}\Delta t$$

Which you've seen before…

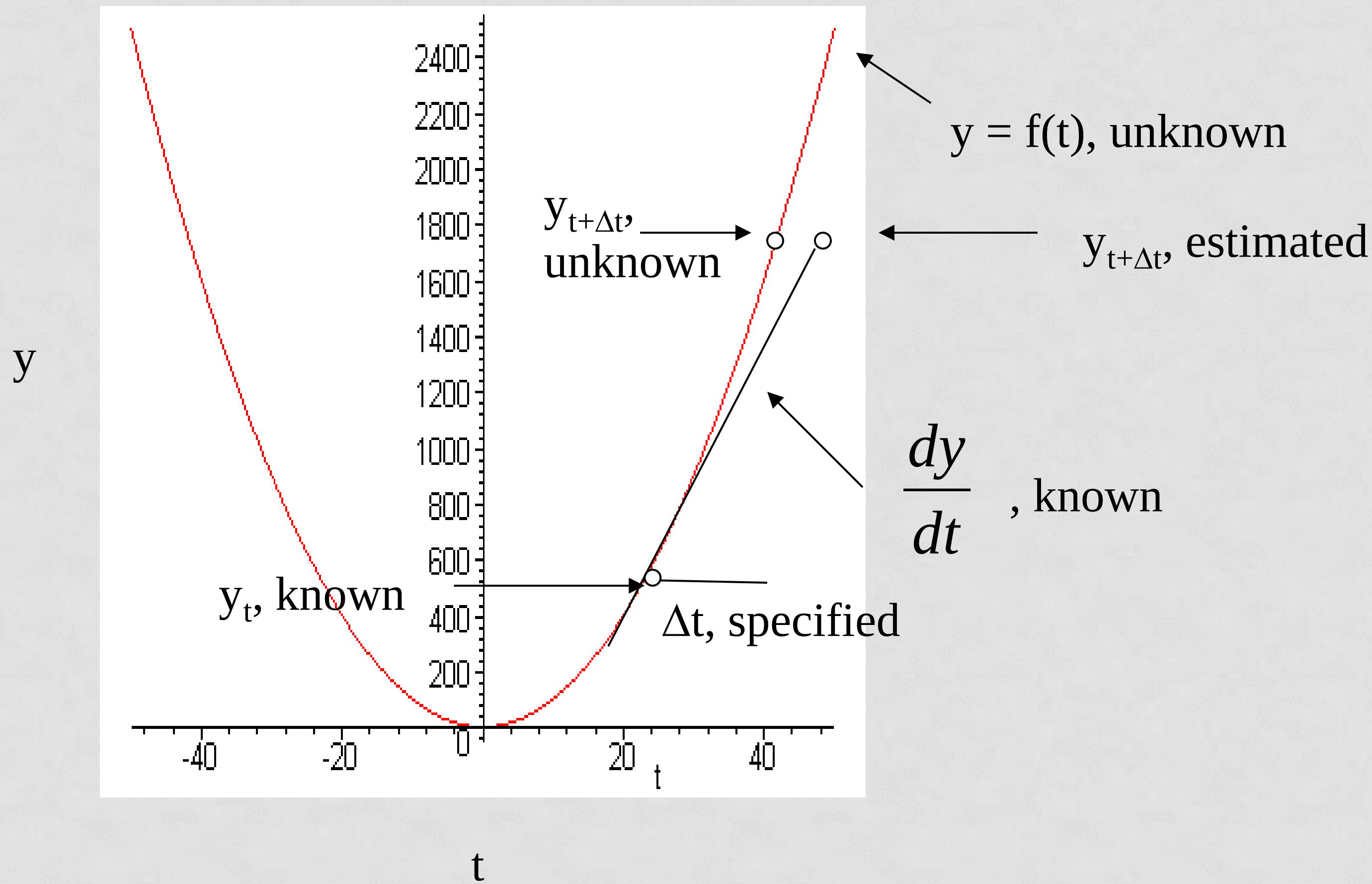
Relationship between continuous and discrete time models

*You used this relationship in Lab 1 to program the logistic rate equation in Visual Basic:

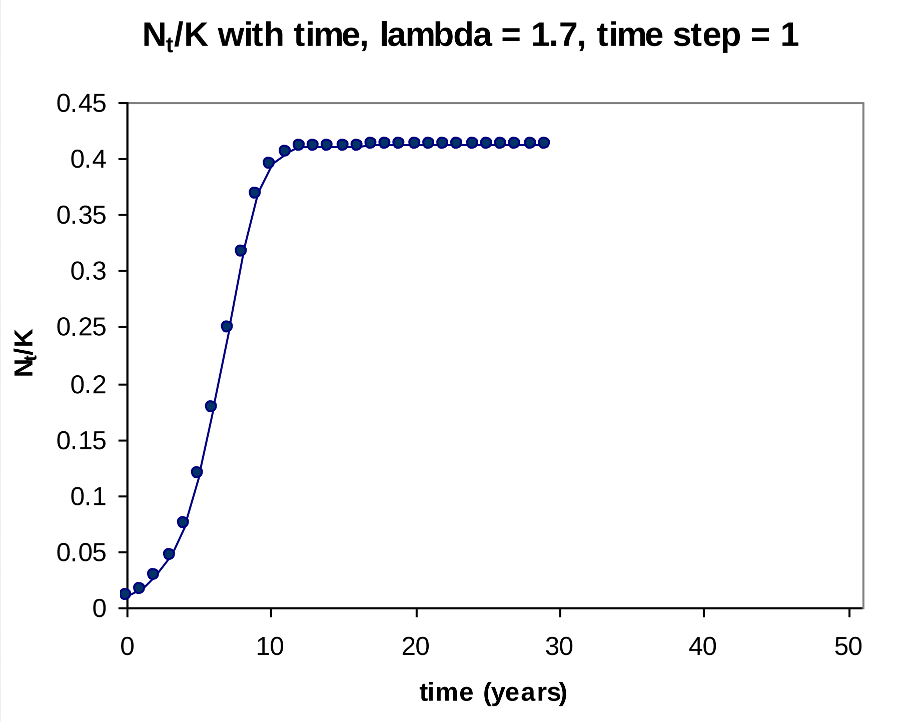
$$N_{t+1} = N_t + rN_t\left(1 - \frac{N_t}{K}\right)\Delta t, \quad \text{where } \Delta t = 1$$

Fundamental Approach of Numerical Integration

$$y_{t+\Delta t} \approx y_t + \frac{dy}{dt}\Delta t$$



$$N_{t+\Delta t} = N_t + \underbrace{rN_t\left(1 - \frac{N_t}{K}\right)}_{dN/dt}\Delta t, \quad \text{where } \Delta t = 1$$



Calculate dN/dt*1 at N_t
Add it to N_t to estimate $N_{t+\Delta t}$

→

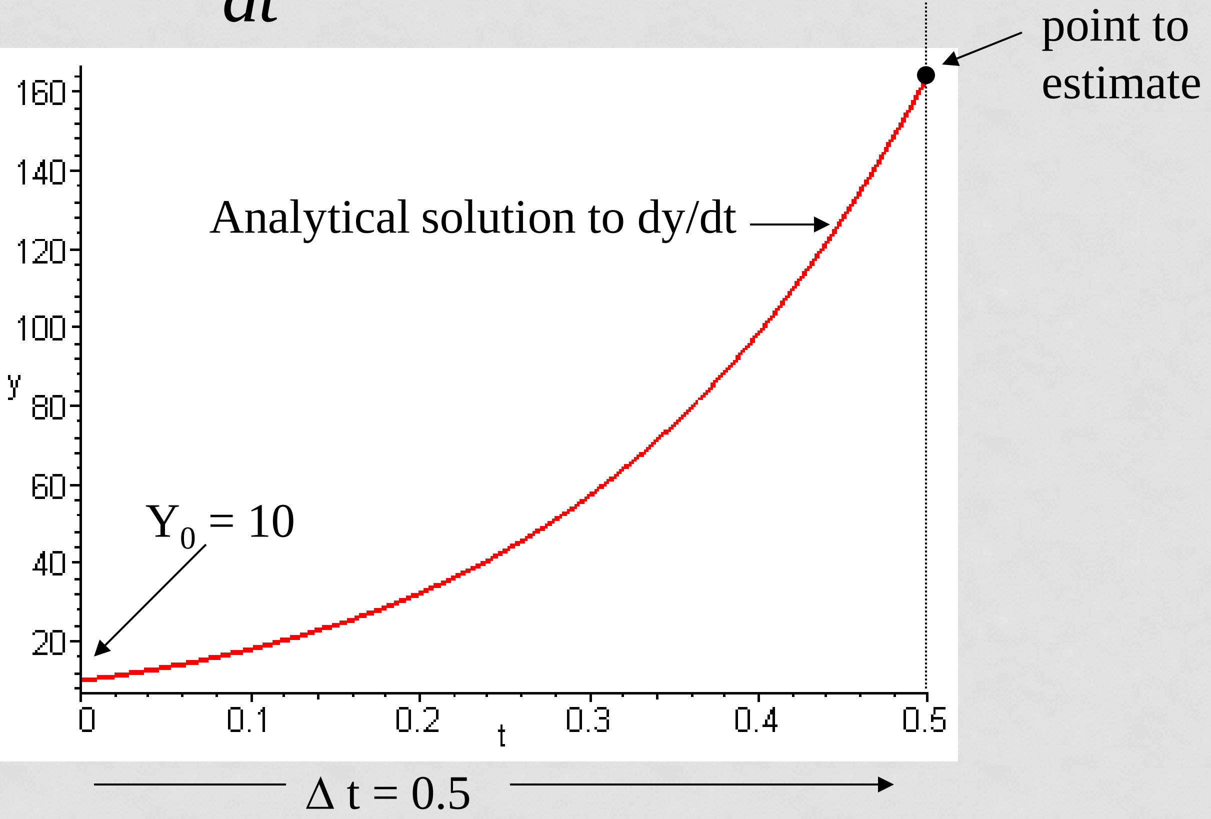
$N_{t+\Delta t}$ becomes the new N_t
Calculte dN/dt * 1 at new N_t
Use dN/dt to estimate next $N_{t+\Delta t}$

Repeat these steps to estimate the state function over your desired time length (here 30 years)

Euler's Method: $y_{t+\Delta t} \approx y_t + {}^{dy}/_{dt}\,\Delta t$

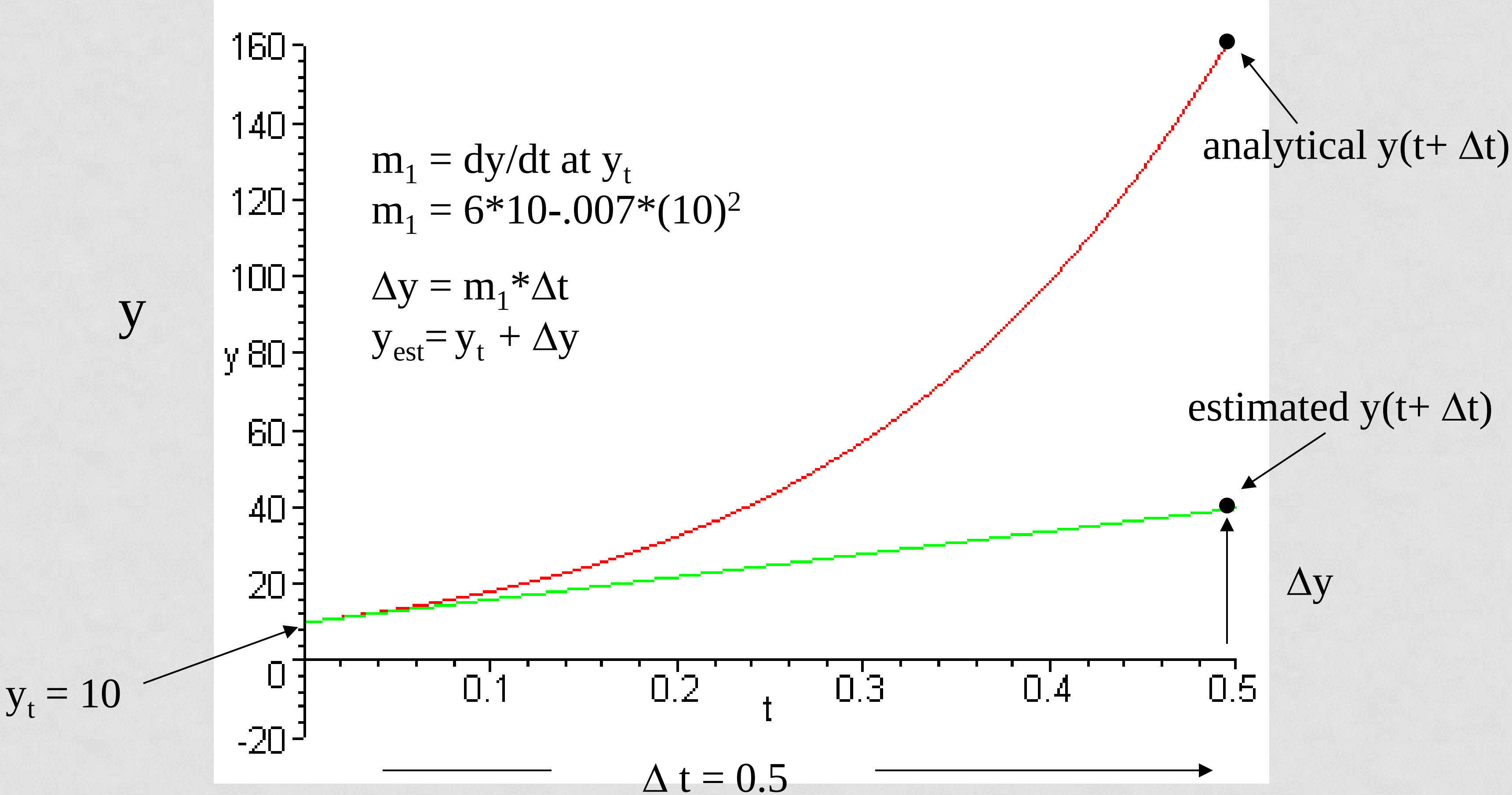
EXAMPLE OF NUMERICAL INTEGRATION

$$\frac{dy}{dt} = 6y - .007y^2$$



Euler's Method: $y_{t+\Delta t} \approx y_t + {}^{dy}/_{dt}\,\Delta t$

$$\frac{dy}{dt} = 6y - .007y^2$$



RUNGE-KUTTA METHODS

MOTIVATION

- We seek accurate methods to solve ODEs that do not require calculating high order derivatives.
- The approach is to use a formula involving unknown coefficients then determine these coefficients to match as many terms of the Taylor series expansion.

SECOND ORDER RUNGE-KUTTA METHOD

$$K_1 = h\ f(x_i, y_i)$$

$$K_2 = h\ f(x_i + \alpha h,\ \ y_i + \beta K_1)$$

$$y_{i+1} = y_i + w_1 K_1 + w_2 K_2$$

Problem :

Find $\alpha, \beta,\ w_1, w_2$

such that y_{i+1} is as accurate as possible.

TAYLOR SERIES IN ONE VARIABLE

The n^{th} order Taylor Series expansion of *f(x)*

$$f(x+h) = \sum_{i=0}^{n} \frac{h^i}{i!} f^{(i)}(x) \quad + \quad \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(\bar{x})$$

Approximation

Error

$$where\ \bar{x}\ is\ between\ x\ and\ x+h$$

DERIVATION OF 2ND ORDER RUNGE-KUTTA METHODS – 1 OF 5

Second Order Taylor Series Expansion

Used to solve ODE: $\dfrac{dy}{dx} = f(x, y)$

$$y_{i+1} = y_i + h\frac{dy}{dx} + \frac{h^2}{2}\frac{d^2y}{dx^2} + O(h^3)$$

which is written as:

$$y_{i+1} = y_i + h\, f(x_i, y_i) + \frac{h^2}{2} f'(x_i, y_i) + O(h^3)$$

DERIVATION OF 2ND ORDER RUNGE-KUTTA METHODS – 2 OF 5

where $f'(x,y)$ is obtained by chain - rule differentiation

$$f'(x,y) = \frac{\partial f(x,y)}{\partial x} + \frac{\partial f(x,y)}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f(x,y)$$

Substituting :

$$y_{i+1} = y_i + f(x_i, y_i)h + \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f(x_i, y_i)\right)\frac{h^2}{2} + O(h^3)$$

TAYLOR SERIES IN TWO VARIABLES

$$f(x+h, y+k) = f(x,y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} \right) +$$

$$\frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + k^2\frac{\partial^2 f}{\partial y^2} + 2hk\frac{\partial^2 f}{\partial x\,\partial y} \right) + \ldots$$

$$= \sum_{i=0}^{n} \frac{1}{i!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^{i} f(x,y) + \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^{n+1} f(\bar{x},\bar{y})$$

approximation *error*

$(\bar{x},\bar{y})$ is on the line joining between (x,y) and $(x+h, y+k)$

DERIVATION OF 2ND ORDER RUNGE-KUTTA METHODS – 3 OF 5

Problem : *Find* α, β, w_1, w_2 such that

$$K_1 = h\ f(x_i, y_i)$$

$$K_2 = h\ f(x_i + \alpha h,\ \ y_i + \beta K_1)$$

$$y_{i+1} = y_i + w_1 K_1 + w_2 K_2$$

Substituting :

$$y_{i+1} = y_i + w_1 h\ f(x_i, y_i) + w_2 h\ f(x_i + \alpha h,\ y_i + \beta K_1)$$

DERIVATION OF 2ND ORDER RUNGE-KUTTA METHODS – 4 OF 5

$$f(x_i + \alpha h,\ y_i + \beta K_1) = f(x_i, y_i) + \alpha h \frac{\partial f}{\partial x} + \beta K_1 \frac{\partial f}{\partial y} + \ldots$$

Substituting :

$$y_{i+1} = y_i + w_1 h\ f(x_i, y_i) + w_2 h \left(f(x_i, y_i) + \alpha h \frac{\partial f}{\partial x} + \beta K_1 \frac{\partial f}{\partial y} + \ldots \right)$$

$$y_{i+1} = y_i + (w_1 + w_2) h\ f(x_i, y_i) + w_2 h \left(\alpha h \frac{\partial f}{\partial x} + \beta K_1 \frac{\partial f}{\partial y} + \ldots \right)$$

$$y_{i+1} = y_i + (w_1 + w_2) h\ f(x_i, y_i) + w_2 \alpha h^2 \frac{\partial f}{\partial x} + w_2 \beta h^2 \frac{\partial f}{\partial y} f(x_i, y_i) + \ldots$$

DERIVATION OF 2ND ORDER RUNGE-KUTTA METHODS – 5 OF 5

We derived two expansions for y_{i+1}:

$$y_{i+1} = y_i + (w_1 + w_2)h\, f(x_i, y_i) + w_2 \alpha h^2 \frac{\partial f}{\partial x} + w_2 \beta h^2 \frac{\partial f}{\partial y} f(x_i, y_i) + \ldots$$

$$y_{i+1} = y_i + f(x_i, y_i)h + \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f(x_i, y_i) \right) \frac{h^2}{2} + O(h^3)$$

Matching terms, we obtain the following three equations:

$$w_1 + w_2 = 1, \qquad w_2 \alpha = \frac{1}{2}, \quad \text{and} \quad w_2 \beta = \frac{1}{2}$$

3 equations with 4 unknowns $\Rightarrow$ infinite solutions

One possible solution: $\alpha = \beta = 1, \quad w_1 = w_2 = \dfrac{1}{2}$

2^{ND} ORDER RUNGE-KUTTA METHODS

$$K_1 = h\ f(x_i, y_i)$$

$$K_2 = h\ f(x_i + \alpha h,\quad y_i + \beta K_1)$$

$$y_{i+1} = y_i + w_1 K_1 + w_2 K_2$$

Choose α, β, w_1, w_2 such that:

$$w_1 + w_2 = 1,\ w_2 \alpha = \frac{1}{2},\ \text{and}\ w_2 \beta = \frac{1}{2}$$

ALTERNATIVE FORM

Second Order Runge Kutta

$$K_1 = h\ f(x_i, y_i)$$

$$K_2 = h\ f(x_i + \alpha h,\ \ y_i + \beta K_1)$$

$$y_{i+1} = y_i + w_1 K_1 + w_2 K_2$$

Alternative Form

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \alpha h,\ \ y_i + \beta h k_1)$$

$$y_{i+1} = y_i + h\left(w_1 k_1 + w_2 k_2\right)$$

CHOOSING α, β, W_1 AND W_2

$$\text{For example, choosing } \alpha = 1, \text{ then } \beta = 1, \; w_1 = w_2 = \frac{1}{2}$$

Second Order Runge - Kutta method becomes :

$$K_1 = h \; f(x_i, y_i)$$

$$K_2 = h \; f(x_i + h, y_i + K_1)$$

$$y_{i+1} = y_i + \frac{1}{2}\left(K_1 + K_2\right) = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y_{i+1}^0)\right)$$

This is *Heun's Method* with a Single Corrector

CHOOSING α, β, W_1 AND W_2

$$\text{Choosing } \alpha = \frac{1}{2} \text{ then } \beta = \frac{1}{2}, \quad w_1 = 0, \; w_2 = 1$$

Second Order Runge - Kutta method becomes :

$$K_1 = h \; f(x_i, y_i)$$

$$K_2 = h \; f(x_i + \frac{h}{2}, y_i + \frac{K_1}{2})$$

$$y_{i+1} = y_i + K_2 = y_i + h \; f(x_i + \frac{h}{2}, y_i + \frac{K_1}{2})$$

This is the Midpoint Method

2ND ORDER RUNGE-KUTTA METHODS ALTERNATIVE FORMULAS

$$\alpha w_2 = \frac{1}{2}, \quad \beta w_2 = \frac{1}{2}, \quad w_1 + w_2 = 1$$

$$\text{Pick any nonzero } \alpha \text{ number}: \ \beta = \alpha, \ w_2 = \frac{1}{2\alpha}, \quad w_1 = 1 - \frac{1}{2\alpha}$$

Second Order Runge Kutta Fomulas (select $\alpha \neq 0$)

$$K_1 = h\, f(x_i, y_i)$$

$$K_2 = h\, f(x_i + \alpha h, \quad y_{\mathrm{i}} + \alpha K_1)$$

$$y_{i+1} = y_i + \left(1 - \frac{1}{2\alpha}\right) K_1 + \frac{1}{2\alpha} K_2$$

SECOND ORDER RUNGE-KUTTA METHOD
EXAMPLE

Solve the following system to find $x(1.02)$ using RK2

$$\dot{x}(t) = 1 + x^2 + t^3, \qquad x(1) = -4,\ h = 0.01,\ \alpha = 1$$

STEP 1:

$$K_1 = h\ f(t_0 = 1, x_0 = -4) = 0.01(1 + x_0{}^2 + t_0{}^3) = 0.18$$

$$K_2 = h\ f(t_0 + h, x_0 + K_1)$$

$$= 0.01(1 + (x_0 + 0.18)^2 + (t_0 + .01)^3) = 0.1662$$

$$x(1 + 0.01) = x(1) + \left(K_1 + K_2\right)/2$$

$$= -4 + (0.18 + 0.1662)/2 = -3.8269$$

SECOND ORDER RUNGE-KUTTA METHOD
EXAMPLE

STEP 2

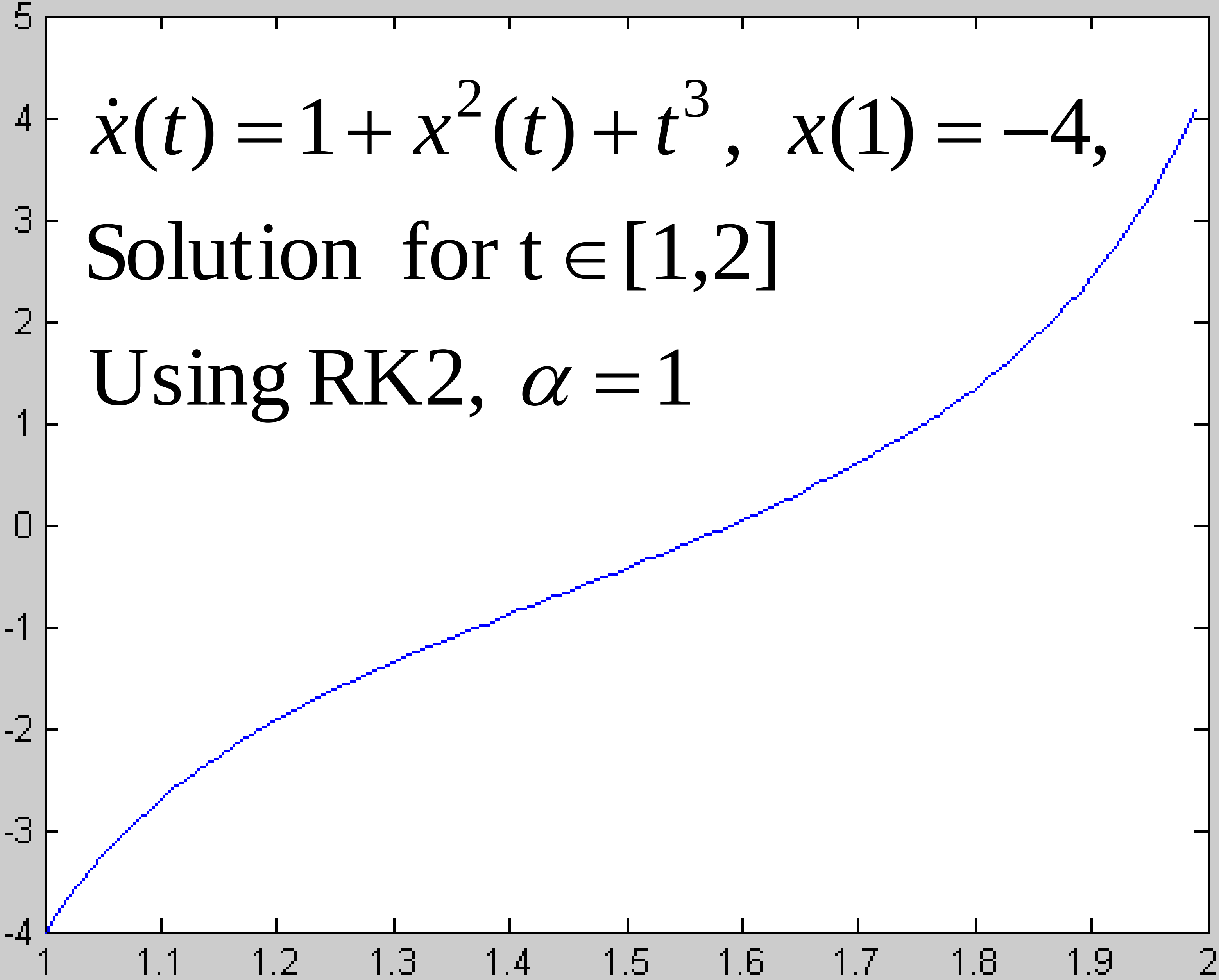
$$K_1 = h\ f(t_1 = 1.01, x_1 = -3.8269) = 0.01(1 + {x_1}^2 + {t_1}^3) = 0.1668$$

$$K_2 = h\ f(t_1 + h, x_1 + K_1)$$

$$= 0.01(1 + (x_1 + 0.1668)^2 + (t_1 + .01)^3) = 0.1546$$

$$x(1.01 + 0.01) = x(1.01) + \frac{1}{2}\left(K_1 + K_2\right)$$

$$= -3.8269 + \frac{1}{2}(0.1668 + 0.1546) = -3.6662$$



$\dot{x}(t) = 1 + x^2(t) + t^3, \ x(1) = -4,$
Solution for $t \in [1,2]$
Using RK2, $\alpha = 1$
5
4
3
2
1
0
-1
-2
-3
-4
1
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2

2ND ORDER RUNGE-KUTTA

RK2

Typical value of $\alpha = 1$, Know as RK2

Equivalent to Heun's method with a single corrector

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h, y_i + k_1 h)$$

$$y_{i+1} = y_i + \frac{h}{2}(k_1 + k_2)$$

Local error is $O(h^3)$ and global error is $O(h^2)$

HIGHER-ORDER RUNGE-KUTTA

Higher order Runge-Kutta methods are available.

Derived similar to second-order Runge-Kutta.

Higher order methods are more accurate but require more calculations.

3RD ORDER RUNGE-KUTTA

RK3

Know as RK3

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \frac{h}{2},\; y_i + \frac{1}{2} k_1 h)$$

$$k_3 = f(x_i + h,\; y_i - k_1 h + 2k_2 h)$$

$$y_{i+1} = y_i + \frac{h}{6}\left(k_1 + 4k_2 + k_3\right)$$

Local error is $O(h^4)$ and Global error is $O(h^3)$

4^{TH} ORDER RUNGE-KUTTA

RK4

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{h}{2},\ \ y_i + \frac{1}{2}k_1h\right)$$

$$k_3 = f\left(x_i + \frac{h}{2},\ \ y_\mathrm{i} + \frac{1}{2}k_2h\right)$$

$$k_4 = f(x_i + h,\ \ y_\mathrm{i} + k_3h)$$

$$y_{i+1} = y_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

Local error is $O(h^5)$ and global error is $O(h^4)$

HIGHER-ORDER RUNGE-KUTTA

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{4}h,\ y_i + \frac{1}{4}k_1 h\right)$$

$$k_3 = f\left(x_i + \frac{1}{4}h,\ y_i + \frac{1}{8}k_1 h + \frac{1}{8}k_2 h\right)$$

$$k_4 = f\left(x_i + \frac{1}{2}h,\ y_i - \frac{1}{2}k_2 h + k_3 h\right)$$

$$k_5 = f\left(x_i + \frac{3}{4}h,\ y_i + \frac{3}{16}k_1 h + \frac{9}{16}k_4 h\right)$$

$$k_6 = f\left(x_i + h,\ y_i - \frac{3}{7}k_1 h + \frac{2}{7}k_2 h + \frac{12}{7}k_3 h - \frac{12}{7}k_4 h + \frac{8}{7}k_5 h\right)$$

$$y_{i+1} = y_i + \frac{h}{90}\left(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6\right)$$

EXAMPLE
4^{TH}-ORDER RUNGE-KUTTA METHO

RK4

$$\frac{dy}{dx} = 1 + y + x^2$$

$$y(0) = 0.5$$

$$h = 0.2$$

Use RK4 to compute $y(0.2)$ *and* $y(0.4)$

EXAMPLE: RK4

Problem:

$$\frac{dy}{dx} = 1 + y + x^2, \qquad y(0) = 0.5$$

Use $RK4$ *to find* $y(0.2), y(0.4)$

4^{TH} ORDER RUNGE-KUTTA

RK4

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \frac{h}{2}, \quad y_i + \frac{1}{2}k_1 h)$$

$$k_3 = f(x_i + \frac{h}{2}, \quad y_\text{i} + \frac{1}{2}k_2 h)$$

$$k_4 = f(x_i + h, \quad y_\text{i} + k_3 h)$$

$$y_{i+1} = y_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

Local error is $O(h^5)$ and global error is $O(h^4)$

EXAMPLE: RK4

See RK4 Formula

Problem:

$$\frac{dy}{dx} = 1 + y + x^2, \qquad y(0) = 0.5$$

Use RK4 to find $y(0.2), y(0.4)$

$$h = 0.2$$

$$f(x, y) = 1 + y + x^2$$

$$x_0 = 0, \quad y_0 = 0.5$$

Step 1

$$k_1 = f(x_0, y_0) = (1 + y_0 + x_0{}^2) = 1.5$$

$$k_2 = f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1 h) = 1 + \left(y_0 + 0.15\right) + \left(x_0 + 0.1\right)^2 = 1.64$$

$$k_3 = f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2 h) = 1 + \left(y_0 + 0.164\right) + \left(x_0 + 0.1\right)^2 = 1.654$$

$$k_4 = f(x_0 + h, y_0 + k_3 h) = 1 + \left(y_0 + 0.16545\right) + \left(x_0 + 0.2\right)^2 = 1.7908$$

$$y_1 = y_0 + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = 0.8293$$

EXAMPLE: RK4

Problem :

$$\frac{dy}{dx} = 1 + y + x^2, \qquad y(0) = 0.5$$

Use RK 4 to find $y(0.2), y(0.4)$

$$h = 0.2$$

$$f(x, y) = 1 + y + x^2$$

$$x_1 = 0.2, \quad y_1 = 0.8293$$

Step 2

$$k_1 = f(x_1, y_1) = 1.7893$$

$$k_2 = f\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1 h\right) = 1.9182$$

$$k_3 = f\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2 h\right) = 1.9311$$

$$k_4 = f(x_1 + h, y_1 + k_3 h) = 2.0555$$

$$y_2 = y_1 + \frac{0.2}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = 1.2141$$

EXAMPLE: RK4

Problem :

$$\frac{dy}{dx} = 1 + y + x^2, \qquad y(0) = 0.5$$

$Use\ RK4\ to\ find\ \ y(0.2), y(0.4)$

Summary of the solution

x_i	y_i
0.0	0.5
0.2	0.8293
0.4	1.2141

SUMMARY

- Runge Kutta methods generate an accurate solution without the need to calculate high order derivatives.
- Second order RK have local truncation error of order $O(h^3)$ and global truncation error of order $O(h^2)$.
- Higher order RK have better local and global truncation errors.
- N function evaluations are needed in the N^{th} order RK method.