Finding and Using Taylor Series

Summary Formulae

Taylor Polynomials at $x=a$

$$P(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Maclaurin series = Taylor series at $x = 0$. Basic Maclaurin series:

1 $$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

2 $$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}.$$

3 $$e^x = 1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

Formulae 1 – 3 can be used for all x.

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Finding Taylor Series (1)

Problem Find the Maclaurin series of the function $\sin\left(x^2\right)$.

Solution Start with the Taylor series $\sin(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1}$.

Substitute $z = x^2$ to get

$$\sin\left(x^2\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(x^2\right)^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{4k+2}.$$

Finding Taylor Series (2)

Problem Find the Maclaurin series of the function $\frac{\sin(x)}{x}$.

Solution Start with the Maclaurin series $\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$.

Divide all terms by x to get

$$\frac{\sin(x)}{x} = \frac{1}{x} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k}.$$

Finding Taylor Series (3)

Problem Find the first three non-zero terms for Maclaurin series of the function $f(x)=\cos(\sin(x))$.

Solution We have $\sin(x)=x-\frac{1}{3!}x^3+\frac{1}{5!}x^5+\ldots$ and $\cos(u)=1-\frac{1}{2!}u^2+\frac{1}{4!}u^4+\ldots$.

Use the shown beginnings of the Taylor series and substitute

$u=x-\frac{1}{3!}x^3+\frac{1}{5!}x^5$ in the Taylor polynomial for $\cos(u)$.

One gets

$$\cos(\sin(x))\approx 1-\frac{1}{2!}\left(x-\frac{1}{3!}x^3+\frac{1}{5!}x^5\right)^2+\frac{1}{4!}\left(x-\frac{1}{3!}x^3+\frac{1}{5!}x^5\right)^4+\text{higher order terms}$$

$$=1-\frac{1}{2!}x^2+\left(\frac{1}{2!}\frac{2}{3!}+\frac{1}{4!}\right)x^4+\text{higher order terms}$$

$$=1-\frac{1}{2!}x^2+\frac{5}{4!}x^4+\text{higher order terms}$$

Error Estimates

For alternating Taylor or Maclaurin series, use the error estimates for alternating series.

Assume that there is a constant L such that for all positive integers k and for all t between 0 and x:

$$\left|f^{(k)}(t)\right| < L.$$

This number L usually depends on x.

Error when Approximating the Function f with its Taylor polynomial of degree m

$$E_m(x) = f(x) - f(0) - \frac{f'(0)}{1!}x - \frac{f''(0)}{2!}x^2 - \ldots - \frac{f^{(m)}(0)}{m!}x^m = f(x) - \sum_{n=0}^{m} \frac{f^{(n)}(0)}{n!}x^n$$

Error Estimate

$$\left|E_m(x)\right| \le L\frac{|x|^{m+1}}{(m+1)!}$$

Taylor Series and Limits (1)

Problem The functions f, g and h satisfy the following:

$$f(3)=g(3)=h(3)=0,\ \ f'(3)=h'(3)=0,\ \ g'(3)=10 \text{ and } f''(3)=5,$$

$$g''(3)=7,\ \ h''(3)=10.$$ Determine the limits $\lim_{x\to 3}\frac{f(x)}{g(x)}$ and $\lim_{x\to 3}\frac{f(x)}{h(x)}$.

Solution The properties of the functions f, g and h imply that their Taylor polynomials of degree 2 at $x=3$ are

$$P_f(x)=\frac{5}{2!}(x-3)^2,\ \ P_g(x)=10(x-3)+\frac{7}{2!}(x-3)^2, \text{ and } P_h(x)=\frac{10}{2!}(x-3)^2.$$

Hence

$$\frac{f(x)}{g(x)}=\frac{\frac{5}{2!}(x-3)^2+\left(\text{terms divisible by } (x-3)^3\right)}{10(x-3)+\frac{7}{2!}(x-3)^2+\left(\text{terms divisible by } (x-3)^3\right)}$$

$$=\frac{\frac{5}{2!}(x-3)+\left(\text{terms containing powers of } (x-3)\right)}{10+\frac{7}{2!}(x-3)+\left(\text{terms containing powers of } (x-3)\right)}\xrightarrow[x\to 3]{}0.$$

Taylor Series and Limits (2)

Problem

The functions f, g and h satisfy the following:

$f(3)=g(3)=h(3)=0,\ f'(3)=h'(3)=0,\ g'(3)=10$ and $f''(3)=5,$

$g''(3)=7,\ h''(3)=10.$ Determine the limits $\lim_{x\to 3}\frac{f(x)}{g(x)}$ and $\lim_{x\to 3}\frac{f(x)}{h(x)}$.

Solution (part b)

Using the Taylor series expansions at $x=3$ one gets

$$\frac{f(x)}{h(x)}=\frac{\frac{5}{2!}(x-3)^2+\left(\text{terms divisible by }(x-3)^3\right)}{\frac{10}{2!}(x-3)^2+\left(\text{terms divisible by }(x-3)^3\right)}$$

$$=\frac{\frac{5}{2!}+\left(\text{terms containing powers of }(x-3)\right)}{\frac{10}{2!}+\left(\text{terms containing powers of }(x-3)\right)}\xrightarrow[x\to 3]{}\frac{1}{2}.$$

Comparison of Functions (1)

Problem Let $f(x) = \sin(x)$, $g(x) = e^x - 1$, and $h(x) = \frac{1}{\sqrt{1-x^2}} - 1.$

Decide which of the above functions takes the smallest values and which the largest values for small positive values of *x*.

Solution We solve the problem by comparing the Taylor series at $x = 0$ of the above functions. The smallest power terms of the series determine the behavior of the function near the origin.

The Taylor expansion for the function $\frac{1}{\sqrt{1+z}}$ starts

$$\frac{1}{\sqrt{1+z}} = (1+z)^{-\frac{1}{2}} = 1 - \frac{1}{2}z + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}z^2 + \dots .$$

Substituting $z = -x^2$ one gets

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \dots .$$

Solution continues

Comparison of Functions (2)

Solution continues

We have $\dfrac{1}{\sqrt{1-x^2}}-1=\dfrac{1}{2}x^2+\dfrac{3}{8}x^4+\dots$.

Taylor series for the other functions are basic Taylor series:

$\sin(x)=x-\dfrac{1}{3!}x^3+\dots$ and

$$e^x-1=\left(1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\dots\right)-1=x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\dots\ .$$

For values of x near 0 it suffices to look at the Taylor polynomials of degree 2.

We have $\sin(x)\approx x$, $e^x-1\approx x+\dfrac{1}{2!}x^2$ and $\dfrac{1}{\sqrt{1-x^2}}-1\approx\dfrac{1}{2}x^2$

Since for small positive values of x, $\dfrac{1}{2}x^2<x<x+\dfrac{1}{2!}x^2$, we deduce

that for small positive values of x, $\dfrac{1}{\sqrt{1-x^2}}-1<\sin(x)<e^x-1$

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